tors of the $1 - D$ system are found as

$$\hat{m}_k(n) = \sqrt{\frac{2}{N+1}} \cos \frac{\pi(k+1)(2n+1)}{2(N+1)}, \quad n = 0, \cdots, N,\ k = 0, \cdots, N-1. \tag{7}$$

The input and output eigenvectors of the $1 + D$ system are obtained by multiplying (2) and (7) with $(-1)^n$. The eigenvalues of the $1 + D$ system are the same as the $1 - D$ system given in (3).

An N-dimensional $1 - D^2$ channel, N even, can be considered as two time-multiplexed $N/2$-dimensional $1 - D$ channels. Consequently, the eigenvalues are in pair equal to

$$\phi_k^2 = 1 - \cos \frac{\pi(k+1)}{(N/2)+1}, \qquad k = 0, \cdots, (N/2) - 1. \tag{8}$$

The two eigenvectors corresponding to a pair of eigenvalues are of the general form $\alpha_1 m_k(2n) + \alpha_2 m_k(2n+1)$, where $\alpha_1^2 + \alpha_2^2 = 1$ and $m_k(n)$ is the eigenvector of the $1 - D$ channel given in (2).

The product of the nonzero eigenvalues of C is equal to

$$\prod_{k=0}^{N-1} \phi_k^2 = |C^t C|, \tag{9}$$

where $|C^t C|$ is the determinant of $C^t C$. This product is an important parameter of the systems based on C. For example, in the transmission system shown in Fig. (1), the volume of the Voronoi region around each constellation point at the channel input is proportional to $(\prod_k \phi_k)^{-1}$ and the required energy is proportional to $(\prod_k \phi_k)^{-2/N}$.

For the $1 \pm D$ channels, assuming $|C^t C| = 2^{-N} \times A_N$ and expanding the determinant, we obtain $A_N = A_{N-1} + 1$. Solving this recursive equation with the initial value $A_1 = 2$ results in $A_N = N + 1$. Consequently, for the $1 \pm D$ channels, we have

$$\prod_{k=0}^{N-1} \phi_k^2 = 2^{-N} \times (N+1). \tag{10}$$

For the $1 - D^2$ channel, we have

$$\prod_{k=0}^{N-1} \phi_k^2 = 2^{-N} \times [(N/2)+1]^2. \tag{11}$$

For all three channels, modulation with the input eigenvectors can be performed by using the even discrete sine transform. For modulation with the output eigenvectors, we can use the block diagram shown in Fig. (2). Using (7), modulation with the output eigenvectors can also be achieved using an $N + 1$-point even discrete cosine transform. In this case, samples of the modulating vector are shifted by one sample and filled with zero. Ref. [3] shows how both the discrete sine transform and the discrete cosine transform can be efficiently calculated.

III. Summary

The input and output eigenvectors and the eigenvalues of the $1 \pm D$ and $1 - D^2$ systems are calculated. The product of the nonzero eigenvalues are found in closed form. In all cases, the multiplication by the input or output eigenvectors can be achieved by using fast transform algorithms.

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Number of Nearest Neighbors in a Euclidean Code

Kenneth Zeger and Allen Gersho

***Abstract*—A Euclidean code is a finite set of points in n-dimensional Euclidean space $\mathscr{R}^n$. The total number of nearest neighbors of a given codepoint in the code is called its touching number. We show that the maximum number of codepoints F_n that can share the same nearest-neighbor codepoint is equal to the maximum kissing number τ_n in n dimensions, that is, the maximum number of unit spheres that can touch a given unit sphere without overlapping. We then apply a known upper bound on τ_n to obtain $F_n \le 2^{n(0.401 + o(1))}$, which improves upon the best known upper bound of $F_n \le 2^{n(1 + o(1))}$. We also show that the average touching number T of all the points in a Euclidean code is upper bounded by τ_n.**

I. Introduction

A *Euclidean code* is a finite set Y of $M > 1$ points in n-dimensional Euclidean space $\mathscr{R}^n$. A vector quantizer codebook and a code (or signal constellation) for the Gaussian channel are both examples of Euclidean codes. In both cases, the nearest-neighbor partition (also known as the *Voronoi* partition) of the space induced by the code is of particular importance in evaluating the performance of the code. For vector quantizers, a source vector is encoded by identifying in which region of the partition it lies. For Gaussian channels, a selected codepoint is corrupted by an additive Gaussian noise vector and the maximum *a posteriori* decoder identifies in which region of the Voronoi partition the resulting vector lies.

A special case of a Euclidean code is a *uniform code* (e.g., a lattice code), defined by the property that every codepoint has the same nearest-neighbor distance, $d_{\min}$. Each point of a uniform code can be viewed as the center of a sphere of radius $r_0 = d_{\min}/2$ so that each sphere is contained in the closure of a nearest-neighbor region.

The nearest-neighbor region (or Voronoi cell) of a given point α in a Euclidean code is the set of points in $\mathscr{R}^n$ closer to α than to any other codepoint. This region is a convex set bounded by faces of dimension $n - 1$ that are subsets of hyperplanes. Each such hyperplane is defined as the locus of points equidistant from α and some other codepoint β.

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There are two important types of neighboring points for convex polytopal partitions of space, namely, adjacent points and touching points. Each codepoint that defines a face of the polytopal cell containing α is called an *adjacent point* of α. For any point α in the code Y, we say that a point β in Y is a *touching point* (or *nearest neighbor*) of α, if β is a closest point of α in the code Y. In this case,

$$d_\alpha \doteq \min\{\|\alpha - \gamma\|: \gamma \in Y, \gamma \neq \alpha\} = \|\alpha - \beta\|, \qquad (1)$$

where we call d_α the *touching distance* of the point α. Note that every touching point of a given codepoint is also an adjacent point to it, but the converse does not necessarily hold.

Some interesting properties of adjacent points are known. Using Euler's formula for planar maps, one can show that the average number of adjacent points per codepoint for any Euclidean code in two dimensions is at most 6. Klee [1] used a construction based on the existence of simplicial neighborly d-polytopes, to show that for $n \geq 4$ and for any M, there always exists a Euclidean code such that every point in the code has exactly $M - 1$ adjacent points. It has been shown for tree-structured Euclidean codes that the average number of adjacent points for $n = 2$ is less than 4, and for $n \geq 3$ this number cannot be universally upper bounded by a constant (i.e., independent of the number of points), by showing that there exist families of codes such that the average number of adjacent points grows linearly with the size of the code [2]. However, much less is known about touching points for Euclidean codes.

Let $d_{\min}$ be the *minimum distance* between any pair of points in the code. Thus

$$d_{\min} \doteq \{\|\alpha - \beta\|: \alpha, \beta \in Y, \alpha \neq \beta\}. \qquad (2)$$

The minimum distance of a code plays a central role in obtaining error probability bounds for the Gaussian channel under conditions of high signal-to-noise ratio. The number of nearest neighbors of each codepoint also plays an important role in determining the performance of a code.

Since $M > 1$ for any code, every point in the code has at least one touching point and at most $M - 1$ touching points. The *touching number* of a codepoint α is the number of touching points of α. The *total touching number* of a code is the sum of the touching numbers of each point in the code. The *average touching number* T is the total touching number divided by M (the size of the code). Also of interest for a given codepoint α is the total number of codepoints which have α as a touching point. Denote by F_n the maximum number of points in $\mathscr{R}^n$ that can share a common nearest neighbor, where the maximum is taken over all possible arrangements of points.

In general, the touching number may differ from one codepoint to another. Also, note that a codepoint might not be the nearest neighbor of any other codepoint. It is simple to construct a code in which each point has only one touching point. For example, in $\mathscr{R}$, let $\alpha_k = 1/k$ for $k = 1, 2, \cdots, M$. Then α_{k+1} is the unique touching point of α_k for $k = 1, 2, \cdots, M-1$, and α_{M-1} is the unique touching point of α_M. It is also possible for a codepoint to have $M - 1$ touching points. For example, let α_1 be the origin and let all remaining points $\alpha_2, \cdots, \alpha_M$ lie on the surface of a sphere in n dimensions ($n \geq 2$), centered at the origin.

For M large enough and for a fixed dimension n, it is not possible for all codepoints to simultaneously have $M - 1$ touching points, nor is it possible for a codepoint to be a nearest neighbor of arbitrarily many other codepoints. We shall demonstrate that there is, in fact, an upper bound on both F_n and the average touching number T of a code. This upper bound will be independent of the size M of the code.

II. Main Result

A sphere packing in any dimension is a collection of disjoint unit radius open spheres. In a sphere packing, the *kissing number* (or *contact number*) of any sphere is the number of spheres in the packing that it is tangent to. In a lattice packing, each sphere has the same kissing number. The *maximum kissing number* in $\mathscr{R}^n$, denoted by τ_n, is the largest kissing number that can be attained by any packing of n-dimensional spheres.

The maximum kissing number is known exactly in only very few dimensions, namely, $\tau_1 = 2$, $\tau_2 = 6$, $\tau_3 = 12$, $\tau_8 = 240$, and $\tau_{24} = 196\,560$. It is known that for some n, no lattice packing can achieve the maximum kissing number. On the other hand, the 24-dimensional kissing number is attained in the well-known Leech lattice. Good upper and lower bounds on τ_n are known for low dimensions and are summarized in [3]. For arbitrarily high-dimensional space, the following lemma gives the best known upper and lower bounds on τ_n. The upper bound is due to Kabatiansky and Levenshtein [4] and the lower bound is due to Wyner [5].

Lemma 1: The maximum kissing number in n-dimensional space is bounded as

$$2^{0.2075n(1+o(1))} \leq \tau_n \leq 2^{0.401n(1+o(1))}. \qquad (3)$$

Throughout this paper, the notation $o(1)$ will refer to the asymptotics of the dimension n, and will be independent of the code size M. In most applications of Euclidean codes, the rate, $R = (\log_2 M)/n$ (bits per dimension), of the code exceeds 1, so that usually $M = 2^{Rn} \gg \tau_n$.

The main results of this paper (Theorem 1 and Theorem 2) show that τ_n equals the maximum number F_n of codepoints that can share a common nearest-neighbor codepoint and that τ_n upper bounds the average touching number T of a code. Since M is often used as an upper bound to the average number of nearest neighbors of a code, this shows that the tighter upper bound τ_n can be used advantageously. For a uniform code, the touching number for any point cannot exceed τ_n and hence the average touching number T is bounded by τ_n. Theorem 2 shows that for nonuniform codes, the average touching number has the same upper bound.

The number of points in $\mathscr{R}^n$ that can share a common nearest-neighbor point plays an important role in a widely diversified set of fields of research. For example, some researchers in psychology have investigated the nearest-neighbor problem from a statistical point of view [6], [7]. In [8], an extensive experimental study was done to compute a histogram for the values of F_n based on a statistical model. Other studies of the number of points having a given point as a nearest neighbor have been done in such fields as sociology, biology, cognitive psychology, and ecology (e.g., see the references listed in [6]).

The quantity F_n also plays an important role in the field of nonparametric (distribution-free) estimation. In [9], it is shown that F_n is independent of the code size for metrics induced by arbitrary norms in $\mathscr{R}^n$. A bound of $F_n \leq 3^n - 1 \approx 2^{1.585n}$ for all $n \geq 1$ was cited in [10] as an upper bound which is independent of the code size. Rogers [11, Th. 3] derived bounds on the number of unit spheres needed to cover a given sphere of arbitrary radius when $n \geq 9$. Fritz [12], citing the bounds of Rogers, noted that F_n can be approximately upper bounded by $F_n \leq 2^{n(1+o(1))}$. Stone [13, Prop. 12] has shown that F_n can be upper bounded by the minimum number of 60° cones emanating from a point that can cover space. Combining Stone's result with the earlier result of Rogers also gives $F_n \leq 2^{n(1+o(1))}$.

By Lemma 1, the kissing number can be upper bounded by $\tau_n \le 2^{n(0.401+o(1))}$, making our bound, $F_n \le 2^{n(0.401+o(1))}$, a significant improvement in tightness over the previously known bounds mentioned above. The essential difference between our bound and the weaker one of Stone and Rogers is that theirs is based on a *minimal covering* while ours is based on a *maximal packing*.

Theorem 1: The maximum number of points in $\mathscr{R}^n$ which can have a common nearest neighbor is equal to the maximum kissing number (i.e., $F_n = \tau_n$), and is thus bounded as $2^{0.2075n(1+o(1))} \le F_n \le 2^{n(0.401+o(1))}$.

Proof: Consider any set of τ_n nonoverlapping spheres which are tangent to a common sphere. The centers of these τ_n spheres each have the center of the common sphere as a common nearest neighbor. Hence $F_n \ge \tau_n$ and it remains to show $F_n \le \tau_n$.

Suppose there exist $k > \tau_n$ points $\alpha_1, \cdots, \alpha_k$ which have the common touching point β. Let $d_m = \min\{d_{\alpha_i}: 1 \le i \le k\}$, where $d_{\alpha_i} = \|\alpha_i - \beta\|$ for each $i = 1, 2, \cdots, k$, and define

$$\gamma_i = \frac{d_{\alpha_i} - d_m}{d_{\alpha_i}}\beta + \frac{d_m}{d_{\alpha_i}}\alpha_i. \tag{4}$$

It is easily seen that

$$\|\gamma_i - \beta\| = \frac{d_m}{d_{\alpha_i}}\|\alpha_i - \beta\| = d_m. \tag{5}$$

The points $\gamma_1, \cdots, \gamma_k$ lie on a sphere in $\mathscr{R}^n$ of radius d_m and centered at β. Therefore, in the code $\{\beta, \gamma_1, \gamma_2, \cdots, \gamma_k\}$, each γ_i is a touching point of β with touching distance d_m.

We now show that for all i and j, if $i \ne j$, then $\|\gamma_i - \gamma_j\| \ge d_m$. Without loss of generality, suppose that i and j are indexes such that

$$\|\alpha_i - \beta\| \le \|\alpha_j - \beta\| \le \|\alpha_j - \alpha_i\|. \tag{6}$$

Then

$$\begin{aligned} 2(\alpha_i - \beta)\cdot(\alpha_j - \beta) &= \|\alpha_i - \beta\|^2 + \|\alpha_j - \beta\|^2 - \|\alpha_i - \alpha_j\|^2 \\ &\le d_{\alpha_i}^2 \le d_{\alpha_i}d_{\alpha_j}. \end{aligned} \tag{7}$$

But

$$2(\gamma_i - \beta)\cdot(\gamma_j - \beta) = \frac{2d_m^2}{d_{\alpha_i}d_{\alpha_j}}(\alpha_i - \beta)\cdot(\alpha_j - \beta) \tag{8}$$

by the definition of γ_i and γ_j.

Hence,

$$2(\gamma_i - \beta)\cdot(\gamma_j - \beta) \le d_m^2 \tag{9}$$

and

$$\begin{aligned} \|\gamma_i - \gamma_j\|^2 &= \|\gamma_i - \beta\|^2 + \|\gamma_j - \beta\|^2 \\ &\quad - 2(\gamma_i - \beta)\cdot(\gamma_j - \beta) \\ &\ge d_m^2 + d_m^2 - d_m^2 = d_m^2. \end{aligned} \tag{10}$$

Hence, in the code $\{\beta, \gamma_1, \gamma_2, \cdots, \gamma_k\}$, the point β has touching number k that exceeds τ_n. However, then k nonoverlapping spheres of radius $d_m/2$ could be placed about each touching point of β, violating the kissing number bound for the sphere centered at β of radius $d_m/2$. This completes the proof. The required bounds on F_n then follow from Lemma 1. □

Theorem 2: The average touching number of any Euclidean code in $\mathscr{R}^n$ is less than or equal to the maximum kissing number (i.e., $T \le \tau_n$), and thus is upper bounded by $2^{n(0.401+o(1))}$.

Proof: Given a Euclidean code, the directed *nearest-neighbor graph* associated with the code is defined in the following manner. Let each vertex correspond to a particular codepoint. A directed edge goes from vertex α to vertex β if β is a touching point of α. (If α is also a touching point of β, then there is another directed edge from β to α.)

From Theorem 1, it follows that the nearest-neighbor graph of any Euclidean code with $M > 1$ codepoints has the property that each vertex has at most τ_n incoming edges. Hence the total number of edges in the graph cannot exceed $M\tau_n$ (at most linear in the number of vertices) and consequently the total number of outgoing edges from all vertices is also at most $M\tau_n$. Therefore the average touching number T, being the average number of outgoing edges from a vertex, is bounded above by τ_n. The upper bound then follows from Lemma 1. □

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Malfunction in the Peterson–Gorenstein–Zierler Decoder

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***Abstract*—Most versions of the Peterson–Gorenstein–Zierler (PGZ) decoding algorithm are not true bounded distance decoding algorithms in the sense that when a received vector is not in the decoding sphere of**

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