# Asymptotic Capacity of Two-Dimensional Channels With Checkerboard Constraints

Zsigmond Nagy and Kenneth Zeger, *Fellow, IEEE*

***Abstract*—A checkerboard constraint is a bounded measurable set $S \subset \boldsymbol{R}^2$, containing the origin. A binary labeling of the $\boldsymbol{Z}^2$ lattice satisfies the checkerboard constraint $S$ if whenever $t \in \boldsymbol{Z}^2$ is labeled $1$, all of the other $\boldsymbol{Z}^2$-lattice points in the translate $t + S$ are labeled $0$. Two-dimensional channels that only allow labelings of $\boldsymbol{Z}^2$ satisfying checkerboard constraints are studied. Let $A(S)$ be the area of $S$, and let $A(S) \to \infty$ mean that $S$ retains its shape but is inflated in size in the form $\alpha S$, as $\alpha \to \infty$. It is shown that for any open checkerboard constraint $S$, there exist positive reals $K_1$ and $K_2$ such that as $A(S) \to \infty$, the channel capacity $C_S$ decays to zero at least as fast as $(K_1 \log_2 A(S))/A(S)$ and at most as fast as $(K_2 \log_2 A(S))/A(S)$. It is also shown that if $S$ is an open convex and symmetric checkerboard constraint, then as $A(S) \to \infty$, the capacity decays exactly at the rate $4\delta(S)(\log_2 A(S))/A(S)$, where $\delta(S)$ is the packing density of the set $S$. An implication is that the capacity of such checkerboard constrained channels is asymptotically determined only by the areas of the constraint and the smallest (possibly degenerate) hexagon that can be circumscribed about the constraint. In particular, this establishes that channels with square, diamond, or hexagonal checkerboard constraints all asymptotically have the same capacity, since $\delta(S) = 1$ for such constraints.**

***Index Terms*—Asymptotics, constrained channel coding, optical data storage, run-length-limited codes.**

## I. Introduction

One-dimensional channels satisfying run-length constraints are important in magnetic recording applications and two-dimensional channels satisfying run-length constraints have been considered in relation to optical recording applications (see the references in [14]). One-dimensional $(d, k)$ run-length constraints require that in any binary sequence, there be at least $d$ $0$'s between consecutive $1$'s, and the longest run of $0$'s be of length at most $k$.Two-dimensional run-length constraints require that one-dimensional run-length constraints be satisfied both horizontally and vertically in a two-dimensional rectangular binary array.

An important special two-dimensional channel is one satisfying the $(d, \infty)$ run-length constraint. In two dimensions, the $(1, \infty)$ constraint, for example, has been studied in terms of computing the channel capacity [4], [7] and for efficient coding algorithms [21], [22]. The capacity of the $(1, \infty)$-constrained channel is not known exactly, but has been very accurately upper- and lower-bounded.

Manuscript received July 13, 2002; revised May 3, 2003. This work was supported in part by the National Science Foundation and the University of California, San Diego Center for Wireless Communications.

The authors are with the Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, CA 92093-0407 USA (e-mail: nagy@code.ucsd.edu; zeger@ucsd.edu).

Communicated by K. Abdel-Ghaffar, Associate Editor for Coding Theory.
Digital Object Identifier 10.1109/TIT.2003.815766

If a two-dimensional $(d, \infty)$ run-length constraint is further constrained along one diagonal direction to similarly only allow $(d, \infty)$-constrained sequences (e.g., in the northwest–southeast direction as shown in Fig. 1 f)), then this is equivalent to a channel that allows binary labeled patterns of a hexagonal grid (as opposed to a rectangular grid) such that a $(d, \infty)$ run-length constraint must be met along the three natural axes of the hexagonal grid. A complicated nonrigorous[1] derivation of the capacity for the case $d = 1$ (known as the "hard hexagon model") was given in [3], from which an analytic expression for the capacity was presented in [15], [19], and [26].

Various interpretations of two-dimensional run-length-constrained capacities appear in other fields of study. For example, the two-dimensional $(1, \infty)$ capacity is equal to the growth rate (as $N \to \infty$) of the number of configurations of mutually nonattacking princes on an $N \times N$ chessboard, where a "prince" acts as a chess piece that can move to any square that shares an edge with its current location. Likewise, the analytic capacity in [15], [19], [26] gives the growth rate of the number of configurations of mutually nonattacking princes on a hexagonal chessboard. The growth rates of the number of certain configurations of mutually nonattacking chess pieces on an $N \times N$ chessboard have been extensively studied (e.g., for kings, in [18], [30]). The capacity calculations in [4] were formulated in terms of counting independent sets of vertices in graphs. The capacities are also closely related to gases, lattices, and Ising model entropies in statistical mechanics [2].

In addition to run-length constraints, other types of constraints can be used to model two-dimensional channels for certain applications [1], [8]–[10], [12], [13], [23]–[25], [27], [28]. For example, run-length constraints along diagonals in both directions (northwest–southeast and northeast–southwest) can be imposed, in addition to horizontal and vertical constraints. An example of a circularly symmetric two-dimensional constraint occurs by requiring that any point in the two-dimensional $\boldsymbol{Z}^2$ lattice be labeled $0$ if it is within a prescribed Euclidean distance from a lattice point with label $1$. In other words, each $1$ must be surrounded by a certain circle of $0$'s.

One could alternatively require that every $1$ be surrounded by $0$'s falling in a given sized hexagon, square, or more generally any other shape of interest. In general, a large class of such two-dimensional constraints can be characterized by some bounded measurable two-dimensional set $S$, and the requirement that for every $1$ stored in the plane, it must at least be sur-

[1]Baxter comments on his derivation [2, p. 409]: "It is not mathematically rigorous, in that certain analyticity properties of $\kappa$ are assumed, and the results of Chapter 13 (which depend on assuming that various large-lattice limits can be interchanged) are used. However, I believe that these assumptions, and therefore (14.1.18)–(14.1.24), are in fact correct."

rounded by a set of 0's arranged in the shape of $S$. Such a code is said to satisfy the constraint $S$. These constraints are known as checkerboard constraints [29]. Two-dimensional $(d, \infty)$ constraints are examples of checkerboard constraints, in which case the set $S$ is the union of the intervals $[-d, d]$ on both the horizontal and vertical axes in the plane (i.e., a "$+$" shape). Likewise, the hexagonal-grid constraint studied in [2] is a checkerboard constraint. It was noted in [29]:

> "For example, in two-dimensional optical recording systems bits may be stored on media in the form of dark or bright patterns. As the storage 'disk' is read, these patterns pass through various lenses and other image-forming devices, thus producing intersymbol interference (ISI). Checkerboard constraints will reduce this ISI, so naturally we wish to analyze such constraints."

In this paper, we focus on the asymptotic behavior of the capacity of two-dimensional channels satisfying checkerboard constraints. In the special case of the two-dimensional $(d, \infty)$ run-length-constrained channel, the asymptotic behavior of the capacity is well understood. It was shown in [16] that the capacity decays to zero at the exact rate $(\log_2 d)/d$ as $d \to \infty$. For a general checkerboard constraint, the asymptotics analogous to run-length constraints are when the constraint $S$ retains its shape but is inflated in size in the form $\alpha S$ as $\alpha \to \infty$.

As $\alpha$ goes to infinity, the amount of information that can be stored per unit area shrinks to zero. In other words, the capacity decays to zero. We determine the rate at which the capacity goes to zero as a function of the area $A(S)$ of the constraint, for certain classes of checkerboard constraints. If the checkerboard constraint $S$ is assumed to be open, then we show (Theorem V.2) that as $A(S) \to \infty$, the capacity decays to zero at a rate bounded between $(K_1 \log_2 A(S))/A(S)$ and $(K_2 \log_2 A(S))/A(S)$, for some positive finite constants $K_1$ and $K_2$. Theorem V.2 makes precise a prediction given in [29]: "Intuitively, we expect that the capacity of a given constraint will be inversely proportional to the number of *zeros* in the constraint." If the checkerboard constraint $S$ is additionally assumed to be convex and symmetric, then we show (Theorem IV.4) that as $A(S) \to \infty$, the capacity decays to zero at the rate $4\delta(S)(\log_2 A(S))/A(S)$, where $\delta(S)$ is the packing density of the set $S$. Thus, for example, since the packing density (in the plane) of squares or convex-symmetric hexagons is $\delta(S) = 1$, this implies that the capacity of two-dimensional channels satisfying square or hexagon checkerboard constraints is asymptotically equal to $4(\log_2 A(S))/A(S)$ as the area grows without bound. Similarly, if $S$ is a circular constraint, then the asymptotic capacity is $\frac{2\pi}{\sqrt{3}}(\log_2 A(S))/A(S)$ since $\delta(S) = \pi/(2\sqrt{3})$.

Since the constraint $S$ corresponding to a two-dimensional $(d, \infty)$ run-length constraint is neither open nor convex, the results in this paper do not specialize to the $(d, \infty)$ constraint case, but they do provide an interesting related checkerboard constraint result.

## II. Preliminaries

Let $\boldsymbol{R}^2$ denote the two-dimensional plane. A two-dimensional *lattice* is a set $T \subset \boldsymbol{R}^2$ of the form $T = \{\kappa u + \lambda v \colon \kappa, \lambda \in \boldsymbol{Z}\}$ where $u, v \in \boldsymbol{R}^2$ are independent. In particular, $\boldsymbol{Z}^2$ denotes the two-dimensional integer lattice. Given a set $S \subset \boldsymbol{R}^2$, a *labeling* of $S$ is any function

$$f \colon S \cap \boldsymbol{Z}^2 \to \{0, 1\}.$$

For any set $S \subset \boldsymbol{R}^2$, let $A(S)$ be the area of $S$ and let

$$\Lambda(S) = \left| S \cap \boldsymbol{Z}^2 \right|$$

be the number of $\boldsymbol{Z}^2$-lattice points contained in $S$.

A set $S \subset \boldsymbol{R}^2$ is *symmetric* if $x \in S \Leftrightarrow -x \in S$. For any $S \subset \boldsymbol{R}^2$, $y \in \boldsymbol{R}^2$, and $\alpha \in \boldsymbol{R}$ let $S + y = \{x + y \colon x \in S\}$ and $\alpha S = \{\alpha x \colon x \in S\}$. Also, for sets $S, T \subset \boldsymbol{R}^2$ let

$$S + T = \{x + y \colon x \in S, y \in T\}.$$

The closure of $S$ is denoted by $\overline{S}$.

For any $a \in \boldsymbol{R}^2$ and $b \in \boldsymbol{R}$, a *line* is a set

$$l = \{x \in \boldsymbol{R}^2 \colon (a \cdot x) + b = 0\}$$

where $a \cdot x$ is the dot product of $a$ and $x$. A line $l$ is a *supporting line* to the set $S \subset \boldsymbol{R}^2$ if $l \cap \overline{S} \neq \emptyset$ and one of the two closed half-planes determined by $l$ contains $S$.

Let $R, S, T \subset \boldsymbol{R}^2$. For each $t \in T$, the set $S + t$ is called a *$T$-translate* of $S$. The set $T \subset \boldsymbol{R}^2$ is called an

- *$S$-packing* of $R$ if the interiors of the $T$-translates are disjoint and are contained in $R$;
- *$S$-covering* of $R$ if the union of the closures of the $T$-translates contains $R$;
- *$S$-tiling* of $R$ if it is both an $S$-packing and an $S$-covering of $R$.

A *rectangle* is any set

$$R_{(\kappa, \lambda)}^{(\mu, \nu)} = \left\{(x, y) \in \boldsymbol{R}^2 \colon \kappa \le x \le \mu, \lambda \le y \le \nu\right\}$$

for some $\kappa, \lambda, \mu, \nu \in \boldsymbol{R}$.

The following definitions are from [20]. Let $S, T \subset \boldsymbol{R}^2$ and define

$$\rho_+(S, T) = \limsup_{\kappa, \lambda, \mu, \nu \to \infty} \rho(S, T, \kappa, \lambda, \mu, \nu)$$
$$\rho_-(S, T) = \liminf_{\kappa, \lambda, \mu, \nu \to \infty} \rho(S, T, \kappa, \lambda, \mu, \nu)$$

where

$$\rho(S, T, \kappa, \lambda, \mu, \nu) = \frac{\sum_{t \in T} A\left((S + t) \cap R_{(-\kappa, -\lambda)}^{(\mu, \nu)}\right)}{A\left(R_{(-\kappa, -\lambda)}^{(\mu, \nu)}\right)}.$$

The *packing density* of $S$ is

$$\delta(S) = \sup_T \rho_+(S, T) \tag{1}$$

where the supremum is taken over all $S$-packings $T$ of $\boldsymbol{R}^2$, and the *covering density* of $S$ is

$$\theta(S) = \inf_T \rho_-(S, T) \tag{2}$$

where the infimum is taken over all $S$-coverings $T$ of $\boldsymbol{R}^2$.

The following lemma states that the densest packing and the sparsest covering using convex symmetric sets are attained by a lattice packing and lattice covering, respectively.

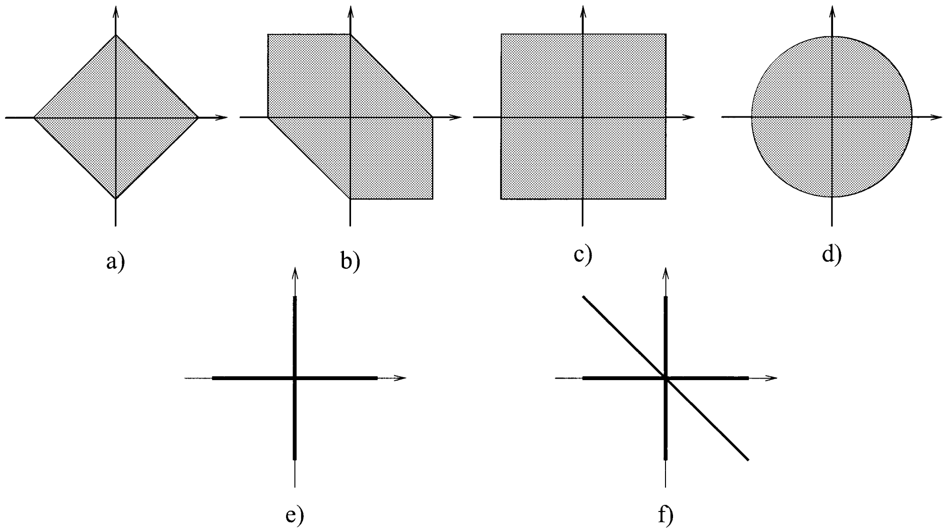


Fig. 1. Various checkerboard constraints: a) diamond; b) hexagon; c) square; d) circle; e) $(d, \infty)$ run length; and e) $(d, \infty)$ hexagonal-grid run length.

*Lemma II.1 [20, pp. 12, 17]:* For every convex-symmetric set $S \subset \boldsymbol{R}^2$ there exist lattices $L_1$ and $L_2$ such that $\delta(S) = \rho_+(S, L_1)$ and $\theta(S) = \rho_-(S, L_2)$.

A two-dimensional *constrained channel* is a set of labelings of $\boldsymbol{R}^2$. Such labelings are called *valid*. A constraint is a description of which labelings are valid for a particular constrained channel. A *checkerboard constraint* is a bounded measurable set $S \subset \boldsymbol{R}^2$ that contains the origin. The terminology "checkerboard constraint" was introduced in [29] to mean a "two-dimensional arrangement of *zeros* that must surround every *one* in a two-dimensional binary code," which is consistent with the present definition.

Given a set $V \subset \boldsymbol{R}^2$ and a checkerboard constraint $S$, a labeling $f$ of $V$ is said to be $S$*-valid* on $V$ if

$$\begin{aligned}&f(y) = 0 \text{ whenever } f(x) = 1,\\ &\forall\, x \in V \cap \boldsymbol{Z}^2\\ &\forall\, y \in (x + S) \cap (V \setminus \{x\}) \cap \boldsymbol{Z}^2.\end{aligned}$$

That is, $f$ satisfies the checkerboard constraint $S$ on the set $V \subset \boldsymbol{R}^2$. Note that any $S$-valid labeling of a subset of $\boldsymbol{R}^2$ can be extended to an $S$-valid labeling of $\boldsymbol{R}^2$ by making the labeling equal 0 outside of the subset. The number of $S$-valid labelings of a set $V \subset \boldsymbol{R}^2$ is denoted by $N_S(V)$. The *capacity* $C_S$ corresponding to the checkerboard constraint $S$ is

$$C_S = \lim_{\kappa, \lambda, \mu, \nu \to \infty} \frac{\log_2 N_S\left(R_{(-\kappa, -\lambda)}^{(\mu, \nu)}\right)}{A\left(R_{(-\kappa, -\lambda)}^{(\mu, \nu)}\right)}. \tag{3}$$

A proof given in [16] shows that the above limit exists.

An example of a checkerboard constraint is a run-length constraint. For each nonnegative integer $d$, the two-dimensional $(d, \infty)$ *run-length constraint* is defined as the following subset of $\boldsymbol{R}^2$:

$$S_{d, \infty} = \{(0, x)\colon -d \le x \le d\} \cup \{(x, 0)\colon -d \le x \le d\}. \tag{4}$$

The capacities of various channels satisfying convex checkerboard constraints were studied in [29]. These included the diamond, hexagonal, square, and $(d, \infty)$ run-length checkerboard constraints, and are shown in Fig. 1.

*Lemma II.2:* Let $V \subset \boldsymbol{R}^2$, let $S$ be a checkerboard constraint, and let $f$ be a labeling of $V$. If $f$ is $S$-valid then $f$ is $-S$-valid.

*Proof:* Suppose $f$ is not $-S$-valid. Then there exist $x, y \in V$ such that $f(x) = f(y) = 1$ and $y \in x + (-S)$. This implies that $x \in y + S$ and, therefore, $f$ is not $S$-valid. $\square$

*Corollary II.3:* Let $V \subset \boldsymbol{R}^2$, $S$ be a checkerboard constraint, and $f$ be a labeling of $V$. Then $f$ is $S$-valid if and only if $f$ is $(S \cup -S)$-valid.

Corollary II.3 follows immediately from Lemma II.2. It follows from Corollary II.3 that every checkerboard constraint $S$ is equivalent to the symmetric checkerboard constraint $S \cup -S$ in the sense that the sets of $S$-valid labelings and $(S \cup -S)$-valid labelings of any set $V \subset \boldsymbol{R}^2$ are identical. That is, any nonsymmetric checkerboard constraint is also a symmetric checkerboard constraint. Therefore,

$$N_S(V) = N_{S \cup -S}(V)$$

for any set $V \subset \boldsymbol{R}^2$, establishing Corollary II.4 below. Thus, no generality is lost if we restrict attention to symmetric checkerboard constraints when computing capacities.

*Corollary II.4:* If $S$ is a checkerboard constraint then $C_S = C_{S \cup -S}$.

*Lemma II.5:* If $S$ is a convex symmetric checkerboard constraint which is either open or closed, and $k$ is a positive integer, then

$$\frac{1}{2}\, S \subset \frac{k+1}{2}\, S + u_1 + \cdots + u_k$$

for any $u_1, \ldots, u_k \in \frac{1}{2}\overline{S}$.

*Proof:* Let $y \in \frac{1}{2}S$. Then

$$\begin{aligned}(k+1)y &\in \frac{k+1}{2}\, S\\ (k+1)u_i &\in \frac{k+1}{2}\, \overline{S}, \qquad \text{for } i = 1, \ldots, k.\end{aligned}$$

Since $S$ is symmetric

$$-(k+1)u_i \in \frac{k+1}{2}\,\overline{S}.$$

The quantity

$$y - \sum_{i=1}^{k} u_i$$

is a convex combination (with weights $\frac{1}{k+1}$) of the points

$$(k+1)y, \, -(k+1)u_1, \, \ldots, \, -(k+1)u_k$$

and $\frac{k+1}{2}S$ is a convex set. If $S$ is open then $y$ lies in the interior of $\frac{1}{2}S$ and thus also in the interior of $\frac{k+1}{2}S$. Therefore, by convexity

$$y - \sum_{i=1}^{k} u_i \in \frac{k+1}{2}\, S$$

(see [17, p. 111, Theorem 5]). If $S$ is closed then

$$y - \sum_{i=1}^{k} u_i \in \frac{k+1}{2}\,\overline{S} = \frac{k+1}{2}\, S.$$

In both cases

$$y = \left(y - \sum_{i=1}^{k} u_i\right) + \sum_{i=1}^{k} u_i \in \frac{k+1}{2}\, S + \sum_{i=1}^{k} u_i. \qquad \square$$

*Lemma II.6:* Let $S$ be a convex symmetric checkerboard constraint which is either open or closed. For any $S$-valid labeling $f$ of $\boldsymbol{R}^2$, any set $Q \subset \frac{1}{2}S$, and every $w \in \boldsymbol{R}^2$, the set $Q + w$ cannot contain more than one $\boldsymbol{Z}^2$-lattice point with label 1.

*Proof:* Suppose to the contrary that there exist $\boldsymbol{Z}^2$-lattice points

$$x, \, y \in Q + w \subset \frac{1}{2} S + w$$

such that

$$f(x) = f(y) = 1.$$

Then

$$x - w, \qquad y - w \in \frac{1}{2} S.$$

Taking $k = 1$ in Lemma II.5 implies that

$$x - w \in S + y - w$$

and, therefore,

$$x \in S + y$$

which contradicts the assumption that $f$ is $S$-valid. $\square$

*Remark II.7:* Suppose $f$ is a valid labeling. In the special case where the set of $\boldsymbol{Z}^2$-lattice points with label 1 forms a lattice, Lemma II.6 follows from Minkowski's Convex Body Theorem [5, pp. 71–72].[2]

[2]There is a typographical error in the last line of the statement of the corollary in [5]. It should read "whose difference $\frac{1}{2}x_1 - \frac{1}{2}x_2$ is in $\Lambda$."

*Lemma II.8:* Let $S$ be an open convex symmetric checkerboard constraint, and let $f$ be a labeling of $\boldsymbol{R}^2$. Then $f$ is $S$-valid if and only if the set

$$f^{-1}(1) = \left\{x \in \boldsymbol{Z}^2\colon \, f(x) = 1\right\}$$

is a $\frac{1}{2}S$-packing of $\boldsymbol{R}^2$.

*Proof:* Suppose $f$ is not $S$-valid. Then there exist distinct $x, \, y \in f^{-1}(1)$ such that $y \in S + x$. Since $S$ contains the origin, $y \in \frac{1}{2}S + y$. If $y \in \frac{1}{2}S + x$, then $(\frac{1}{2}S + x) \cap (\frac{1}{2}S + y) \neq \emptyset$ which implies $f^{-1}(1)$ is not a $\frac{1}{2}S$-packing, since $S$ is open. So assume $y \notin \frac{1}{2}S + x$ and likewise $x \notin \frac{1}{2}S + y$. Let

$$l_0 = \{tx + (1-t)y\colon \, t \in [0, 1]\}$$

denote the line segment between the points $x$ and $y$.

Since $S$ is convex and $x$ lies in the interior of $\frac{1}{2}S + x$, the line segment $l_0$ intersects the boundary of $\frac{1}{2}S + x$ in exactly one point, say $r_1$ (see [17, p. 112, Theorem 9]). Similarly, let $r_2$ be the point where $l_0$ intersects the boundary of $\frac{1}{2}S + y$. By the symmetry of $S$, one gets

$$r_2 = -r_1 + x + y$$

and, therefore,

$$|r_1 - x| = |r_2 - y|.$$

Since $x, \, y \in S + x$ and $S$ is convex, we have $l_0 \subset S + x$. Let $r_3$ be the unique point on the extension of $l_0$ beyond $y$, that intersects the boundary of the set $S + x$. Since $S$ is symmetric and since the line segment connecting $x$ to $r_3$ is contained in $S + x$, we have

$$|l_0| < |x - r_3| = 2|x - r_1| = |r_1 - x| + |r_2 - y|.$$

Consequently, $r_1$ is between the points $r_2$ and $y$ on $l_0$, and hence all points of $l_0$ between $r_1$ and $r_2$ are contained in $(\frac{1}{2}S + x) \cap (\frac{1}{2}S + y)$. Thus, $f^{-1}(1)$ is not a $\frac{1}{2}S$-packing.

Now suppose that $f^{-1}(1)$ is not a $\frac{1}{2}S$-packing of $\boldsymbol{R}^2$. Then there exist $x, \, y \in f^{-1}(1)$ such that $(\frac{1}{2}S + x) \cap (\frac{1}{2}S + y) \neq \emptyset$. If $y \in \frac{1}{2}S + x$, then $f$ is not $S$-valid, so assume $y \notin \frac{1}{2}S + x$ (and, likewise, $x \notin \frac{1}{2}S + y$) and let $l_0$, $r_1$, and $r_2$ be defined as above. Since $\frac{1}{2}S + x$ is convex, there exists a supporting line $l_1$ at the point $r_1$ to the set $\frac{1}{2}S + x$ (see [17, p. 143, Corollary 6]). Similarly, by symmetry, $\frac{1}{2}S + y$ has a supporting line

$$l_2 = -l_1 + x + y$$

at the point $r_2$. The lines $l_1$ and $l_2$ are parallel. Let $P_1$ denote the closed halfplane defined by $l_1$ that contains $\frac{1}{2}S + x$, and let $P_2$ denote the closed halfplane defined by $l_2$ that contains $\frac{1}{2}S + y$. Then, $l_1 \subset P_2$, $l_2 \subset P_1$, and $l_1 \neq l_2$ for, otherwise, $\frac{1}{2}S + x$ and $\frac{1}{2}S + y$ would be disjoint. Since $r_1 \in l_1$ and $r_2 \in l_2$, it follows that $r_1$ is between the points $r_2$ and $y$ on $l_0$, and, therefore,

$$|l_0| < |r_1 - x| + |r_2 - y| = 2|r_1 - x|.$$

This implies that $l_0 \subset S + x$, and hence $y \in S + x$. Thus, $f$ is not $S$-valid. $\square$

## III. Hexagonal Checkerboard Constraints

By a *hexagon* we mean any convex six-sided polygon, where it is possible that more than two vertices are colinear. A checkerboard constraint is *hexagonal* if it is an open, convex, symmetric hexagon. An open regular hexagon is an example of a hexagonal checkerboard constraint. By the definition of a hexagon, the diamond and square constraints shown in Fig. 1 a) and c) are also considered hexagonal checkerboard constraints.

*Notation:* Let $U$ be the set of all checkerboard constraints, $f\colon U \to \boldsymbol{R}$, and $L \in \boldsymbol{R}$. For any $S \in U$, if $A(S) > 0$ then we write

$$\lim_{A(S)\to\infty} f(S) = L$$

to mean that $\lim_{\alpha\to\infty} f(\alpha S) = L$. That is, the set $S$ is inflated without bound by the factor $\alpha$ but retains the same shape. Similarly, if $|S \cap \boldsymbol{Z}^2| \geq 2$ then we write

$$\lim_{\Lambda(S)\to\infty} f(S) = L$$

to mean that $\lim_{\alpha\to\infty} f(\alpha S) = L$.

*Theorem III.1:* If $H$ is a hexagonal checkerboard constraint with capacity $C_H$ and area $A(H)$, then

$$\lim_{A(H)\to\infty} C_H \cdot \frac{A(H)}{\log_2 A(H)} = 4.$$

*Proof:* It follows immediately from Lemmas III.3 and III.4 that follow. □

The proof of the following lemma is an easy exercise left to the reader.

*Lemma III.2:* If $H$ is a hexagonal checkerboard constraint, then there is a lattice $H$-tiling of the plane.

*Lemma III.3:* If $H$ is a hexagonal checkerboard constraint, then

$$\limsup_{A(H)\to\infty} C_H \cdot \frac{A\left(\frac{1}{2}H\right)}{\log_2 A\left(\frac{1}{2}H\right)} \leq 1.$$

*Proof:* Let $\beta \in (0,1)$, and for each $\alpha > 0$ let $H_\alpha = \alpha H$. By Lemma III.2, there exists a $\frac{1}{2}(1-\beta)\overline{H}_\alpha$-tiling $T$ of $\boldsymbol{R}^2$. The set $T$ depends on $\alpha$ and $\beta$. Since

$$\frac{1}{2}(1-\beta)\overline{H}_\alpha \subset \frac{1}{2}H_\alpha$$

and $H$ is open, Lemma II.6 implies that for all $t \in T$ and for each $H_\alpha$-valid labeling of $\boldsymbol{R}^2$, at most one $\boldsymbol{Z}^2$-lattice point in $t + \frac{1}{2}(1-\beta)\overline{H}_\alpha$ has label 1. The number of $H_\alpha$-valid labelings of $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$ is upper-bounded if we independently assign an $H_\alpha$-valid labeling to the $\boldsymbol{Z}^2$-lattice points in each of the closed translates $t+\frac{1}{2}(1-\beta)\overline{H}_\alpha$ that intersects $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$ (labelings of boundary points of translates may be overcounted). Recall that $\Lambda(S)$ counts the number of $\boldsymbol{Z}^2$-lattice points in a set $S$. For each $\alpha > 0$, let

$$\epsilon_\alpha = \sup_{t\in T} \frac{\Lambda\left(\frac{1}{2}(1-\beta)\overline{H}_\alpha + t\right)}{A\left(\frac{1}{2}(1-\beta)\overline{H}_\alpha\right)} - 1.$$

Different translates of $\frac{1}{2}(1-\beta)\overline{H}_\alpha$ from the tiling $T$ may contain different numbers of $\boldsymbol{Z}^2$-lattice points, but $\epsilon_\alpha \to 0$ as $\alpha \to \infty$. From the definition of $\epsilon_\alpha$, we have for all $t \in T$,

$$\Lambda\left(\frac{1}{2}(1-\beta)\overline{H}_\alpha + t\right) \leq A\left(\frac{1}{2}(1-\beta)\overline{H}_\alpha\right)(1+\epsilon_\alpha). \tag{5}$$

Define the sets

$$T_i = \left\{t \in T\colon\ \frac{1}{2}(1-\beta)\overline{H}_\alpha + t \subset R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right\}$$
$$T_b = \left\{t \in T\colon\ \emptyset \neq \left(\frac{1}{2}(1-\beta)\overline{H}_\alpha + t\right) \cap R^{(\mu,\nu)}_{(-\kappa,-\lambda)} \neq \frac{1}{2}(1-\beta)\overline{H}_\alpha + t\right\}$$

and denote their cardinalities as

$$n = |T_i|$$
$$m = |T_b|.$$

The integers $n$ and $m$ count the number of translates of $\frac{1}{2}(1-\beta)\overline{H}_\alpha$ from the tiling $T$ that are contained in the rectangle or partially intersect the rectangle, respectively.

Since for any distinct $t_1,\ t_2 \in T_i$, the sets $\frac{1}{2}(1-\beta)H_\alpha + t_1$ and $\frac{1}{2}(1-\beta)H_\alpha + t_2$ are disjoint, we get the lower bound

$$\begin{aligned} A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right) &\geq \sum_{t\in T_i} A\left(\frac{1}{2}(1-\beta)H_\alpha + t\right) \\ &= n \cdot A\left(\frac{1}{2}(1-\beta)H_\alpha\right) \\ &= n \cdot A\left(\frac{1}{2}H_\alpha\right)(1-\beta)^2. \end{aligned} \tag{6}$$

Since $H$ is open, Lemma II.6 implies that at most one $\boldsymbol{Z}^2$-lattice point in $\frac{1}{2}(1-\beta)\overline{H}_\alpha$ can be labeled 1 in an $H_\alpha$-valid labeling. By independently choosing at most one $\boldsymbol{Z}^2$-lattice point to be labeled with a 1 in each of the $m + nT$-translates of $\frac{1}{2}(1-\beta)\overline{H}_\alpha$ that intersect $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$, we obtain an upper bound on the number of $H_\alpha$-valid labelings of $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$, namely

$$\begin{aligned} &N_{H_\alpha}\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right) \\ &\quad\leq \prod_{t\in T_i\cup T_b}\left(\Lambda\left(\frac{1}{2}(1-\beta)\overline{H}_\alpha + t\right) + 1\right) \\ &\quad\leq \sup_{t\in T}\left(\Lambda\left(\frac{1}{2}(1-\beta)\overline{H}_\alpha + t\right) + 1\right)^{|T_i\cup T_b|} \\ &\quad\leq \left(A\left(\frac{1}{2}(1-\beta)\overline{H}_\alpha\right)(1+\epsilon_\alpha) + 1\right)^{m+n} && (7) \\ &\quad= \left(A\left(\frac{1}{2}H_\alpha\right)(1-\beta)^2(1+\epsilon_\alpha) + 1\right)^{m+n} && (8) \end{aligned}$$

where (7) follows from (5); and (8) follows from $A(\frac{1}{2}H) = A(\frac{1}{2}\overline{H})$. Using (3), the lower bound in (6), and the upper bound in (8), the capacity of the checkerboard constraint $H_\alpha$ is upper-bounded as

$$C_{H_\alpha} \leq \lim_{\kappa,\lambda,\mu,\nu\to\infty} \frac{\log_2\left(A\left(\frac{1}{2}H_\alpha\right)(1-\beta)^2(1+\epsilon_\alpha)+1\right)^{m+n}}{nA\left(\frac{1}{2}H_\alpha\right)(1-\beta)^2} \tag{9}$$

$$= \lim_{\kappa,\lambda,\mu,\nu\to\infty}\left(1+\frac{m}{n}\right)\cdot\left(\frac{\log_2 A\left(\frac{1}{2}H_\alpha\right)}{A\left(\frac{1}{2}H_\alpha\right)(1-\beta)^2} + \frac{\log_2\left((1-\beta)^2(1+\epsilon_\alpha)+\frac{1}{A\left(\frac{1}{2}H_\alpha\right)}\right)}{A\left(\frac{1}{2}H_\alpha\right)(1-\beta)^2}\right) \tag{10}$$

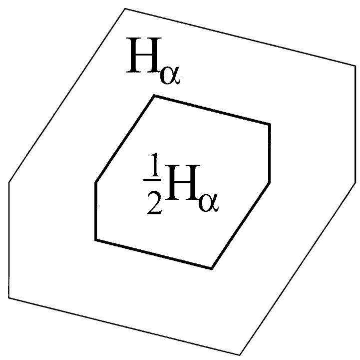


Fig. 2. The checkerboard constraint $H_\alpha$ and its scaled version $\frac{1}{2}H_\alpha$.

$$=\frac{\log_2 A\left(\frac{1}{2}H_\alpha\right)}{A\left(\frac{1}{2}H_\alpha\right)(1-\beta)^2} + \frac{\log_2\left((1-\beta)^2(1+\epsilon_\alpha)+\frac{1}{A\left(\frac{1}{2}H_\alpha\right)}\right)}{A\left(\frac{1}{2}H_\alpha\right)(1-\beta)^2} \tag{11}$$

where the existence of the limit in (9) follows from the existence of the limit in (10); and (11) follows from the fact that $m/n \to 0$ as $\kappa, \lambda, \mu, \nu \to \infty$. Since $\epsilon_\alpha \to 0$ and $A(\frac{1}{2}H_\alpha) \to \infty$ as $\alpha \to \infty$, it follows that

$$\limsup_{\alpha\to\infty} C_{H_\alpha}\cdot\frac{A\left(\frac{1}{2}H_\alpha\right)}{\log_2 A\left(\frac{1}{2}H_\alpha\right)} \le \frac{1}{1-\beta^2}.$$

Since $\beta$ was chosen arbitrarily from the interval $(0, 1)$ we have

$$\limsup_{\alpha\to\infty} C_{H_\alpha}\cdot\frac{A\left(\frac{1}{2}H_\alpha\right)}{\log_2 A\left(\frac{1}{2}H_\alpha\right)} \le \inf_{\beta\in(0,1)}\frac{1}{1-\beta^2}=1. \qquad \square$$

In order to establish a lower bound on $C_H$, we design a labeling algorithm for $\boldsymbol{R}^2$. We again consider translates of inflated copies of $\frac{1}{2}H$ that tile $\boldsymbol{R}^2$, and assign labels to the $\boldsymbol{Z}^2$-lattice points in a block of $\gamma\times\gamma$ neighboring scaled copies of $\frac{1}{2}H$ in the tiling.

*Lemma III.4:* If $H$ is a hexagonal checkerboard constraint, then

$$\liminf_{A(H)\to\infty} C_H\cdot\frac{A\left(\frac{1}{2}H\right)}{\log_2 A\left(\frac{1}{2}H\right)} \ge 1.$$

*Proof:* For each $\alpha > 0$ let $H_\alpha = \alpha H$, and define

$$\epsilon_\alpha = \inf_{t\in\boldsymbol{R}^2}\frac{\Lambda\left(\frac{1}{2}H_\alpha + t\right)}{A\left(\frac{1}{2}H_\alpha\right)} - 1.$$

By Lemma III.2, one can tile $\boldsymbol{R}^2$ with copies of $\frac{1}{2}H_\alpha$ (see Fig. 2) on a lattice. Let $x, y \in \boldsymbol{R}^2$ be independent vectors such that the lattice

$$T = \{ix + jy\colon\ i, j \in \boldsymbol{Z}\}$$

is a $\frac{1}{2}H_\alpha$-tiling. The lattice $T$ depends on $\alpha$. For each positive odd integer $\gamma$, define the sets

$$T_\gamma = \left\{ix+jy\colon -\frac{\gamma-1}{2} \le i, j \le \frac{\gamma-1}{2}\right\}$$
$$B_\gamma = \bigcup_{z\in T_\gamma}\left(z+\tfrac{1}{2}H_\alpha\right)$$

and define $\sigma = 1/(2\gamma)$.

Note that each translate $\frac{1}{2}H_\alpha + z$ (where $z \in T_\gamma$) can be written in the form

$$\frac{1}{2}H_\alpha + z = \frac{1}{2}H_\alpha + ix + jy = \frac{1}{2}H_\alpha + \sum_{l=1}^{2\gamma-2} u_l,$$

where

$$\begin{aligned} -\frac{\gamma-1}{2} &\le i, \qquad j \le \frac{\gamma-1}{2}\\ u_1, \ldots, u_{2i} &= \frac{x}{2}\\ u_{2i+1}, \ldots, u_{2i+2j} &= \frac{y}{2}\\ u_l &= 0, \qquad \text{for all } l > 2i+2j. \end{aligned}$$

Thus,

$$u_l \in \left\{\pm\frac{1}{2}x, \pm\frac{1}{2}y, 0\right\}, \qquad \text{for } l = 1, \ldots, 2\gamma-2.$$

Since $\pm\frac{1}{2}x$ and $\pm\frac{1}{2}y$ lie on the boundary of $\frac{1}{2}H_\alpha$, we have

$$\pm\frac{1}{2}x, \quad \pm\frac{1}{2}y \in \frac{1}{2}\overline{H}_\alpha$$

(and $0 \in \frac{1}{2}\overline{H}_\alpha$). Therefore, by Lemma II.5, we have

$$\frac{1}{2}H_\alpha \subset \frac{2\gamma-1}{2}H_\alpha - \sum_{l=1}^{2\gamma-2}u_l$$

and, therefore,

$$\frac{1}{2}H_\alpha + \sum_{l=1}^{2\gamma-2}u_l \subset \frac{2\gamma-1}{2}H_\alpha \subset \frac{2\gamma}{2}H_\alpha.$$

Thus, $B_\gamma \subset \frac{2\gamma}{2}H_\alpha$ and, hence, $\sigma B_\gamma \subset \frac{1}{2}H_\alpha$.

For all $z, w \in T_\gamma$, define the hexagonal *minicell*

$$D_{z,w} = z + \sigma\left(w + \frac{1}{2}H_\alpha\right).$$

Then

$$\bigcup_{w\in T_\gamma}\sigma\left(w+\frac{1}{2}H_\alpha\right) = \sigma B_\gamma \subset \frac{1}{2}H_\alpha.$$

Thus, for each $z \in T_\gamma$, the minicell $D_{z,z}$ lies inside the hexagonal *cell*

$$z + \frac{1}{2}H_\alpha$$

and is in the same relative position within the $\gamma\times\gamma$ block $z + \sigma B_\gamma$ of minicells in the cell $z + \frac{1}{2}H_\alpha$, as is the position of the cell $z + \frac{1}{2}H_\alpha$ within the $\gamma\times\gamma$ block of cells $B_\gamma$. The vector $w$ determines the position within $z + \frac{1}{2}H_\alpha$ that the minicell lies.

Let $f$ be a labeling of $B_\gamma$ defined as follows. For each $z \in T_\gamma$, label exactly one $\boldsymbol{Z}^2$-lattice point in the minicell $D_{z,z}$ with a 1 and label all other $\boldsymbol{Z}^2$-lattice points in $D_{z,z}$ with a 0. For each

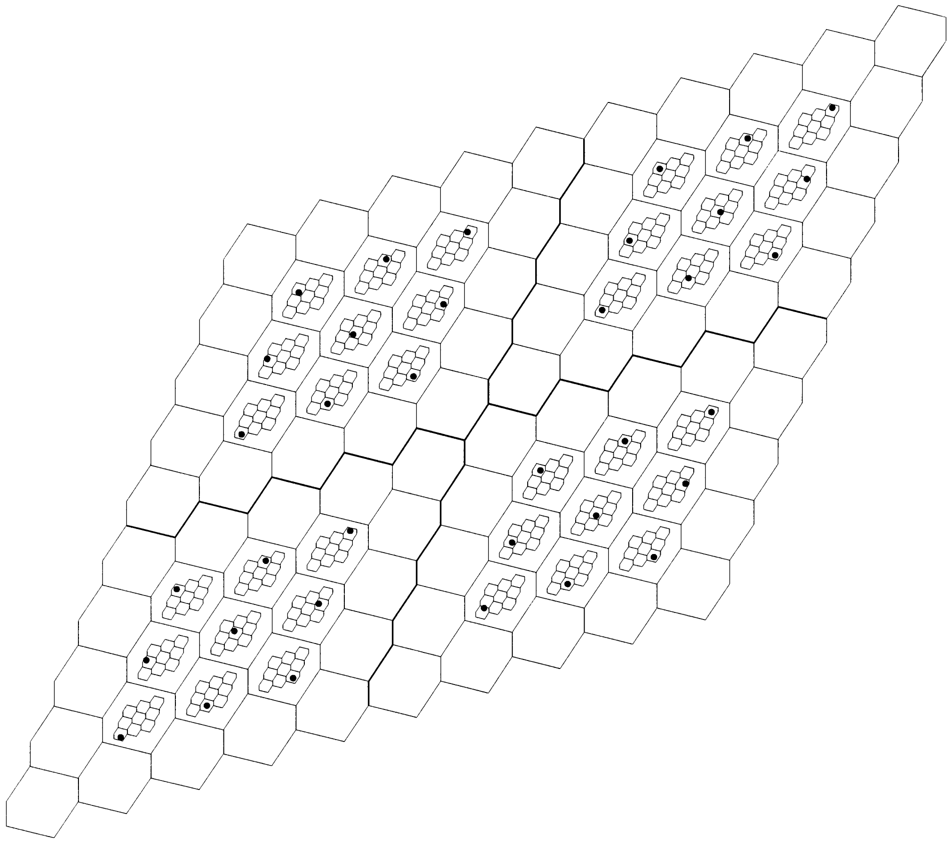

Fig. 3. Four $(\gamma+2)\times(\gamma+2)$ blocks of translates (cells) of $\frac{1}{2}H_\alpha$ for $\gamma = 3$. In each block, the outermost row on each of the four sides has padding cells filled with only 0's. Each nonpadding cell has a $\gamma\times\gamma$ block of minicells in it. Of all the $\boldsymbol{Z}^2$-lattice points in each minicell, only the darkened $\boldsymbol{Z}^2$-lattice points have label 1 in the illustrated labeling. Repeating this construction gives an $H_\alpha$-valid labeling of $\boldsymbol{Z}^2$.

$w,\, z \in T_\gamma$, if $w \neq z$, then label all the $\boldsymbol{Z}^2$-lattice points in the minicell $D_{z,\,w}$ with a 0. Label all other $\boldsymbol{Z}^2$-lattice points with a 0, if they are not in a minicell (i.e., all $\boldsymbol{Z}^2$-lattice points in

$$\left(z+\frac{1}{2}H_\alpha\right)\Big\backslash \bigcup_{w\in T_\gamma} D_{z,\,w}$$

for each $z \in T_\gamma$).

So exactly one $\boldsymbol{Z}^2$-lattice point in each of the $\gamma^2$ cells is labeled 1 and all others are labeled 0. Each such labeling is an $H_\alpha$-valid labeling of $B_\gamma$. In addition, $f$ can be extended to $B_{\gamma+2}$ by labeling every $\boldsymbol{Z}^2$-lattice point in $B_{\gamma+2}\setminus B_\gamma$ with a 0. Then $f$ is an $H_\alpha$-valid labeling of $B_{\gamma+2}$. Fig. 3 illustrates an example labeling.

The total number of such labelings $f$ of $B_\gamma$ is a lower bound to the total number of $H_\alpha$-valid labelings of $B_{\gamma+2}$. That is,

$$\begin{aligned} N_{H_\alpha}(B_{\gamma+2}) &\geq N_{H_\alpha}(B_\gamma)\\ &\geq \prod_{z\in T_\gamma}\Lambda(D_{z,\,z})\\ &\geq \left(\inf_{z\in T}\Lambda(D_{z,\,z})\right)^{|T_\gamma|}\\ &\geq \left(\inf_{z,\,w\in T}\Lambda\left((z+\sigma w)+\frac{\sigma}{2}\,H_\alpha\right)\right)^{\gamma^2}\\ &\geq \left(A\left(\frac{\sigma}{2}\,H_\alpha\right)(1+\epsilon_{\alpha\sigma})\right)^{\gamma^2}\\ &= \left(A\left(\frac{1}{2}H_\alpha\right)\sigma^2(1+\epsilon_{\alpha\sigma})\right)^{\gamma^2} \end{aligned} \tag{12}$$

where we used that fact that $\frac{\sigma}{2}H_\alpha = \frac{1}{2}H_{\alpha\sigma}$. For every $w \in (\gamma+2)T$, let $f_w$ be any such $H_\alpha$-valid labeling of $B_{\gamma+2}\cap\boldsymbol{Z}^2$ and assume its value is 0 elsewhere on $\boldsymbol{Z}^2$. Then an extension to an $H_\alpha$-valid labeling $f$ of all of $\boldsymbol{Z}^2$ can be defined by

$$f(z) = \sum_{w\in(\gamma+2)T} f_w(z-(\gamma+2)w).$$

Although the capacity of a checkerboard constraint is defined in (3) as a limit as the rectangle grows in size, it is straightforward to show that the limit can also be taken over a set such as $B_{\gamma+2}$, as $\gamma$ grows without bound. Thus, since

$$A(B_{\gamma+2}) = (\gamma+2)^2 A\left(\frac{1}{2}H_\alpha\right) \tag{13}$$

the capacity can be lower-bounded using (12) and (13) as

$$\begin{aligned} C_{H_\alpha} &\geq \frac{\log_2 N_{H_\alpha}(B_{\gamma+2})}{A(B_{\gamma+2})}\\ &\geq \frac{\log_2\left(A\left(\frac{1}{2}H_\alpha\right)\sigma^2(1+\epsilon_{\alpha\sigma})\right)^{\gamma^2}}{(\gamma+2)^2 A\left(\frac{1}{2}H_\alpha\right)}\\ &= \left(\frac{\gamma}{\gamma+2}\right)^2\left(\frac{\log_2 A\left(\frac{1}{2}H_\alpha\right)}{A\left(\frac{1}{2}H_\alpha\right)}+\frac{\log_2(\sigma^2(1+\epsilon_{\alpha\sigma}))}{A\left(\frac{1}{2}H_\alpha\right)}\right). \end{aligned}$$

For each $\alpha$, choose

$$\gamma = \lfloor \log_2\alpha \rfloor.$$

Then, as $\alpha \to \infty$, we have

$$\frac{\gamma}{\gamma+2} \to 1$$
$$\alpha\sigma = \frac{\alpha}{2\gamma} \geq \frac{\alpha}{2\log_2 \alpha} \to \infty.$$

Thus, $\epsilon_{\alpha\sigma} \to 0$ as $\alpha \to \infty$. Since

$$\begin{aligned}\frac{-\log_2 \sigma^2}{\log_2 A\left(\frac{1}{2}H_\alpha\right)} &= \frac{-2\log_2 \sigma}{\log_2\left(\alpha^2 A\left(\frac{1}{2}H\right)\right)}\\ &= \frac{2\log_2(2\gamma)}{2\log_2 \alpha + \log_2 A\left(\frac{1}{2}H\right)}\\ &\leq \frac{2 + 2\log_2\log_2 \alpha}{2\log_2 \alpha + \log_2 A\left(\frac{1}{2}H\right)}\\ &\to 0\end{aligned}$$

as $\alpha \to \infty$, we get

$$\liminf_{\alpha\to\infty} C_{H_\alpha} \cdot \frac{A\left(\frac{1}{2}H_\alpha\right)}{\log_2 A\left(\frac{1}{2}H_\alpha\right)} \geq 1. \qquad \square$$

## IV. Open Convex Symmetric Checkerboard Constraints

In this section, we generalize Theorem III.1 to any open convex symmetric checkerboard constraint. The following lemma guarantees that among all minimal-area hexagons containing a given convex symmetric set, at least one is itself also convex and symmetric.

*Lemma IV.1 [6, p. 122]:* Let $S \subset \boldsymbol{R}^2$ be a convex symmetric set. Then there exists a hexagon containing $S$ that is of minimal area, symmetric, and convex.

The following lemma shows that the packing density of a convex symmetric set is achieved by a symmetric circumscribed hexagon of minimal area.

*Lemma IV.2 [20, p. 12]:* Let $S \subset \boldsymbol{R}^2$ be a convex symmetric set and let $H$ be a minimal area symmetric hexagon that contains $S$. Then

$$\delta(S) = \frac{A(S)}{A(H)}.$$

*Lemma IV.3 [11, p. 163]:* Let $R$ be a convex hexagon and $S \subset \boldsymbol{R}^2$ a convex set. The cardinality of any $S$-packing of $R$ is at most $A(R)/A(H)$, where $H$ is a hexagon of least possible area containing $S$.

Note that for $\alpha > 0$, if $H_\alpha$ is a minimal-area symmetric hexagon that contains $\alpha S$, then the ratio $A(\alpha S)/A(H_\alpha)$ is a constant independent of $\alpha$. Thus, if the term $\delta(S)$ appears inside a limit as $A(S) \to \infty$, then the $\delta(S)$ can be brought outside the limit. This fact is used in the proof of Theorem IV.4 that follows.

*Theorem IV.4:* If $S$ is an open convex symmetric checkerboard constraint with area $A(S)$, capacity $C_S$, and packing density $\delta(S)$, then

$$\lim_{A(S)\to\infty} C_S \cdot \frac{A(S)}{\log_2 A(S)} = 4\delta(S). \tag{14}$$

*Proof:* Let $H$ be a symmetric (by Lemma IV.1) hexagon containing $S$, of minimal area $A(H)$. Then

$$\begin{aligned}4 &= \lim_{A(H)\to\infty} C_H \cdot \frac{A(H)}{\log_2 A(H)} &(15)\\ &= \frac{1}{\delta(S)} \cdot \lim_{A(S)\to\infty} C_H \cdot \frac{A(S)}{\log_2 A(S)} &(16)\end{aligned}$$

where (15) follows from Theorem III.1 and (16) follows from Lemma IV.2. Since $S \subset H$, we have $C_S \geq C_H$ and, therefore,

$$\frac{1}{4\delta(S)} \cdot \liminf_{A(S)\to\infty} C_S \cdot \frac{A(S)}{\log_2 A(S)} \geq 1.$$

Since $A(\frac{1}{2}S) = \frac{1}{4}A(S)$, in order to prove the theorem it suffices to show that

$$\frac{1}{\delta(S)} \cdot \limsup_{A(S)\to\infty} C_S \cdot \frac{A\left(\frac{1}{2}S\right)}{\log_2 A\left(\frac{1}{2}S\right)} \leq 1. \tag{17}$$

Let $\beta \in (0, 1)$, and for each $\alpha > 0$, let

$$S_\alpha = \alpha S.$$

We prove (17) by upper-bounding the number of $S_\alpha$-valid labelings of $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$. Let $p$ be the maximum number of $\boldsymbol{Z}^2$-lattice points that can be labeled 1 on $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$ without violating the checkerboard constraint $S_\alpha$. By Lemma II.1, there exists a $\frac{1}{2}(1-\beta)\overline{S}_\alpha$-covering $T$ of $\boldsymbol{R}^2$ that attains $\theta(S)$. Let

$$\begin{aligned}T' &= \left\{t \in T\colon \left(\tfrac{1}{2}(1-\beta)\overline{S}_\alpha + t\right) \cap R^{(\mu,\nu)}_{(-\kappa,-\lambda)} \neq \emptyset\right\}\\ q &= |T'|.\end{aligned}$$

The sets $T$ and $T'$ depend on both $\alpha$ and $\beta$, and the quantities $p$ and $q$ are both functions of $\kappa$, $\lambda$, $\mu$, $\nu$, and $S_\alpha$ ($q$ is also a function of $\beta$). For every $\alpha > 0$, define

$$\epsilon_\alpha = \sup_{t\in\boldsymbol{R}^2} \frac{\Lambda\left(\frac{1}{2}(1-\beta)\overline{S}_\alpha + t\right)}{A\left(\frac{1}{2}(1-\beta)\overline{S}_\alpha\right)} - 1$$

and note that $\epsilon_\alpha \to 0$ as $\alpha \to \infty$. Also, for all $\alpha > 0$ and $t \in T$

$$\begin{aligned}\Lambda\left(\frac{1}{2}(1-\beta)\overline{S}_\alpha + t\right) &\leq (1+\epsilon_\alpha) A\left(\frac{1}{2}(1-\beta)\overline{S}_\alpha\right)\\ &= (1+\epsilon_\alpha)(1-\beta)^2 A\left(\frac{1}{2}S_\alpha\right). &(18)\end{aligned}$$

Since $S$ is open and

$$\frac{1}{2}(1-\beta)\overline{S}_\alpha \subset \frac{1}{2}S_\alpha.$$

Lemma II.6 implies that each of the $q$ copies of $\frac{1}{2}(1-\beta)\overline{S}_\alpha$ intersecting $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$ can contain at most one $\boldsymbol{Z}^2$-lattice point with label 1 in any $S_\alpha$-valid labeling of $\boldsymbol{R}^2$. Thus, $p \leq q$.

The number of $S_\alpha$-valid labelings of $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$ can be upper-bounded by considering all possible collections of $i$ of the $T'$-translates of $\frac{1}{2}(1-\beta)\overline{S}_\alpha$, for $i = 0, \ldots, p$, and assuming that each such translate has exactly one point labeled 1 and no other translate has any points labeled 1. This counts every $S_\alpha$-valid labeling at least once. Since different collections of $i$ of the $T'$-translates might yield the same set of $i$ points being labeled 1, some $S_\alpha$-valid labelings may be counted more than once in this manner. Thus,

$$\begin{aligned} &N_{S_\alpha}\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)\\ &\quad\le \sum_{i=0}^{p}\sum_{\substack{\hat T\subset T'\\ |\hat T|=i}}\prod_{t\in\hat T}\Lambda\left(\frac12(1-\beta)\overline{S}_\alpha+t\right)\\ &\quad\le \sum_{i=0}^{p}\binom{q}{i}\left(\sup_{t\in\boldsymbol{R}^2}\Lambda\left(\frac12(1-\beta)\overline{S}_\alpha+t\right)\right)^i\\ &\quad\le \sum_{i=0}^{p}\binom{q}{i}\left(A\left(\frac12 S_\alpha\right)(1-\beta)^2(1+\epsilon_\alpha)\right)^i \qquad (19)\\ &\quad\le \left(A\left(\frac12 S_\alpha\right)(1-\beta)^2(1+\epsilon_\alpha)\right)^p\sum_{i=0}^{p}\binom{q}{i}\\ &\quad\le \left(A\left(\frac12 S_\alpha\right)(1-\beta)^2(1+\epsilon_\alpha)\right)^p 2^q \qquad (20)\end{aligned}$$

where (19) follows from (18).

Lemma II.8 implies that in any $S_\alpha$-valid labeling, the $\boldsymbol{Z}^2$-lattice points with label 1 in $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$ are a $\frac12 S_\alpha$-packing of $\boldsymbol{R}^2$. By the definition of $p$, there exists an $S_\alpha$-valid labeling of $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$ with exactly $p$ points labeled 1. For the $\frac12 S_\alpha$-packing of $\boldsymbol{R}^2$ determined by the points labeled 1 in this particular labeling, let $p_i$ denote the number of translates of $\frac12 S_\alpha$ that lie inside the boundary of $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$. Then

$$\begin{aligned} &\liminf_{\kappa,\lambda,\mu,\nu\to\infty}\frac{pA\left(\frac12 S_\alpha\right)}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}\\ &\quad=\liminf_{\kappa,\lambda,\mu,\nu\to\infty}\left(\frac{p}{p_i}\right)p_i\cdot\frac{A\left(\frac12 S_\alpha\right)}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}\\ &\quad\le\liminf_{\kappa,\lambda,\mu,\nu\to\infty}\frac{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}{A\left(\frac12 H_\alpha\right)}\cdot\frac{A\left(\frac12 S_\alpha\right)}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)} \qquad (21)\\ &\quad=\frac{A\left(\frac12 S_\alpha\right)}{A\left(\frac12 H_\alpha\right)}=\delta(S_\alpha)=\delta(S) \qquad (22)\end{aligned}$$

where (21) follows from Lemma IV.3 (since the rectangle $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$ is a convex hexagon), and the fact that $p/p_i\to 1$ as $\kappa,\ \lambda,\ \mu,\ \nu\to\infty$; and (22) follows from Lemma IV.2.

Let

$$\begin{aligned} T''&=\left\{t\in T'\colon\ \tfrac12(1-\beta)\overline{S}_\alpha+t\subset R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right\}\\ q_i&=|T''|.\end{aligned}$$

The quantity $q_i$ denotes the number of $T'$-translates of the set $\frac12(1-\beta)\overline{S}_\alpha$ that lie inside the boundary of $R^{(\mu,\nu)}_{(-\kappa,-\lambda)}$. The $\frac12(1-\beta)\overline{S}_\alpha$-covering $T$ of $\boldsymbol{R}^2$ satisfies

$$\begin{aligned} &\liminf_{\kappa,\lambda,\mu,\nu\to\infty}\frac{qA\left(\frac12(1-\beta)\overline{S}_\alpha\right)}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}\\ &\quad=\liminf_{\kappa,\lambda,\mu,\nu\to\infty}\left(\frac{q}{q_i}\right)\frac{q_iA\left(\frac12(1-\beta)\overline{S}_\alpha\right)}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}\\ &\quad=\liminf_{\kappa,\lambda,\mu,\nu\to\infty}\frac{\sum_{t\in T''}A\left(\frac12(1-\beta)\overline{S}_\alpha+t\right)}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)} \qquad (23)\\ &\quad\le\lim_{\kappa,\lambda,\mu,\nu\to\infty}\frac{\sum_{t\in T}A\left(\left(\frac12(1-\beta)\overline{S}_\alpha+t\right)\cap R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}\\ &\quad=\theta\left(\tfrac12(1-\beta)\overline{S}_\alpha\right) \qquad (24)\\ &\quad=\theta(S) \qquad (25)\end{aligned}$$

where (23) follows from the fact that $q/q_i\to 1$ as $\kappa,\ \lambda,\ \mu,\ \nu\to\infty$; and (24) follows from (2). The capacity is then bounded as

$$\begin{aligned} C_{S_\alpha}&\le\liminf_{\kappa,\lambda,\mu,\nu\to\infty}\frac{\log_2\left(\left(A\left(\frac12 S_\alpha\right)(1-\beta)^2(1+\epsilon_\alpha)\right)^p 2^q\right)}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)} \qquad (26)\\ &=\liminf_{\kappa,\lambda,\mu,\nu\to\infty}\left[\frac{p\log_2\left(A\left(\frac12 S_\alpha\right)(1-\beta)^2(1+\epsilon_\alpha)\right)}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}\right.\\ &\qquad\left.+\frac{q}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}\right]\\ &\le\frac{\log_2\left(A\left(\frac12 S_\alpha\right)(1-\beta)^2(1+\epsilon_\alpha)\right)}{A\left(\frac12 S_\alpha\right)}\cdot\delta(S)\\ &\quad+\frac{1}{A\left(\frac12(1-\beta)\overline{S}_\alpha\right)}\cdot\theta(S) \qquad (27)\end{aligned}$$

where (26) follows from (3) and (20); and (27) follows from (22), (25), and the fact that $\epsilon_\alpha\to 0$ as $\alpha\to\infty$. Thus,

$$\begin{aligned} &\limsup_{\alpha\to\infty}C_{S_\alpha}\cdot\frac{A\left(\frac12 S_\alpha\right)}{\log_2 A\left(\frac12 S_\alpha\right)}\\ &\quad\le\delta(S)\cdot\limsup_{\alpha\to\infty}\left(1+\frac{\log_2\left((1-\beta)^2(1+\epsilon_\alpha)\right)}{\log_2 A\left(\frac12 S_\alpha\right)}\right)\\ &\qquad+\theta(S)\cdot\limsup_{\alpha\to\infty}\frac{1}{(1-\beta)^2\log_2 A\left(\frac12 S_\alpha\right)}\\ &\quad=\delta(S). \qquad\square\end{aligned}$$

## V. Arbitrary Checkerboard Constraints

For a given checkerboard constraint $S$, the area $A(S)$ was grown without bound in Theorem IV.4 to obtain convergence rates for the capacity of channels constrained by $S$. As $S$ grows, the area of $S$ becomes approximately equal to the number of

$\boldsymbol{Z}^2$-lattice points in $S$, in the sense that their ratio approaches 1. A larger class of constrained channels may be examined by relaxing the requirement that a constraining set be open and have nonempty interior. However, the area of such a set may be zero, in which case it is more useful to identify the number of internal $\boldsymbol{Z}^2$-lattice points.

The following corollary restates Theorem IV.4 in terms of the number of $\boldsymbol{Z}^2$-lattice points in a constraint instead of the area of a constraint, since both are equal asymptotically as the constraint grows in size. This allows a comparison with two-dimensional run-length-constrained capacities.

*Corollary V.1:* If $S$ is an open convex symmetric checkerboard constraint, then

$$\lim_{\Lambda(S)\to\infty} C_S \cdot \frac{\Lambda(S)}{\log_2 \Lambda(S)} = 4\delta(S)$$

where $\delta(S)$ is the packing density of $S$.

*Proof:* It follows immediately from Theorem IV.4 and the fact that

$$\lim_{A(S)\to\infty} \frac{A(S)}{\Lambda(S)} = 1. \qquad \square$$

The $(d, \infty)$ constraint $S_{d,\infty}$ defined in (4) is a checkerboard constraint but it is neither convex nor open, two properties which were used to obtain Corollary V.1. Furthermore

$$\delta(S_{d,\infty}) = 0.$$

However, a similar result is still true. It is known [16] that the capacity[3] $C_{d,\infty}$ of the two-dimensional $(d, \infty)$ run-length-constrained channel asymptotically decays to zero at the rate $(\log_2 d)/d$. That is,

$$\lim_{d\to\infty} C_{d,\infty} \cdot \frac{d}{\log_2 d} = 1. \tag{28}$$

Since

$$\Lambda(S_{d,\infty}) = 4d + 1$$

for all $d$, the asymptotic capacity in (28) can be written as

$$\lim_{\Lambda(S_{d,\infty})\to\infty} C_{d,\infty} \cdot \frac{\Lambda(S_{d,\infty})}{\log_2 \Lambda(S_{d,\infty})} = 4$$

which is similar in form to Corollary V.1, but is for the nonconvex and nonopen constraint $S_{d,\infty}$.

In fact, a more general rate of convergence can be obtained for the capacity of two-dimensional channels with checkerboard constraints whose interior contains the origin, but without exactly identifying the convergence constant. Such constraints are not necessarily convex. The capacity is shown in Theorem V.2 that follows to still decay asymptotically at the rate $(\log A(S))/A(S)$ in these cases.

*Theorem V.2:* If $S$ is a checkerboard constraint whose interior contains the origin, then

$$0 < \liminf_{A(S)\to\infty} C_S \cdot \frac{A(S)}{\log_2 A(S)} \le \limsup_{A(S)\to\infty} C_S \cdot \frac{A(S)}{\log_2 A(S)} < \infty.$$

[3]The more common notation $C_{d,\infty}$ is used here, instead of the more cumbersome $C_{S_{d,\infty}}$.

*Proof:* Since the origin lies in the interior of $S$, there is an open regular hexagon $R$ contained in $S$ and whose center is the origin. Since $S$ is bounded, it is contained in an open regular hexagon $Q$ whose center is the origin. $R$ and $Q$ are hexagonal checkerboard constraints with packing densities

$$\delta(R) = \delta(Q) = 1.$$

Since $R \subset S \subset Q$, we have

$$1 \le A(R) \le A(S) \le A(Q) < \infty$$
$$C_Q \le C_S \le C_R.$$

Thus, by Theorem III.1

$$\begin{aligned} 4 &= \lim_{\alpha\to\infty} C_Q \cdot \frac{A(\alpha Q)}{\log_2 A(\alpha Q)} \\ &\le \liminf_{\alpha\to\infty} C_S \cdot \frac{A(\alpha Q)}{\log_2 A(\alpha Q)} \\ &= A(Q) \cdot \liminf_{\alpha\to\infty} C_S \cdot \frac{\alpha^2}{\log_2 \alpha^2} \\ &= \frac{A(Q)}{A(S)} \cdot \liminf_{\alpha\to\infty} C_S \cdot \frac{A(\alpha S)}{\log_2 A(\alpha S)} \end{aligned}$$

so that

$$0 < \frac{4A(S)}{A(Q)} \le \liminf_{A(S)\to\infty} C_S \cdot \frac{A(S)}{\log_2 A(S)}.$$

Also, by Theorem III.1

$$\begin{aligned} 4 &= \lim_{\alpha\to\infty} C_R \cdot \frac{A(\alpha R)}{\log_2 A(\alpha R)} \\ &\ge \limsup_{\alpha\to\infty} C_S \cdot \frac{A(\alpha R)}{\log_2 A(\alpha R)} \\ &= A(R) \cdot \limsup_{\alpha\to\infty} C_S \cdot \frac{\alpha^2}{\log_2 \alpha^2} \\ &= \frac{A(R)}{A(S)} \cdot \limsup_{\alpha\to\infty} C_S \cdot \frac{A(\alpha S)}{\log_2 A(\alpha S)} \end{aligned}$$

so that

$$\infty > \frac{4A(S)}{A(R)} \ge \limsup_{A(S)\to\infty} C_S \cdot \frac{A(S)}{\log_2 A(S)}. \qquad \square$$

Note that special cases of Theorem V.2 include when $S$ is an open checkerboard constraint or when $S$ is the closure of an open checkerboard constraint.

## VI. Capacity Relative to a Scaled Lattice

The results obtained in this paper have indicated the asymptotic capacities of certain two-dimensional checkerboard constrained channels. The capacities are given in terms of the "area" of the constraint $S$. The quantity $A(S)$ was defined as the two-dimensional Lebesgue measure of the set $S$. The units of capacity were given as bits per lattice point location on the $\boldsymbol{Z}^2$-lattice, or equivalently, bits per area in the plane. It is reasonable to ask what happens to the results if the lattice itself is scaled. For example, suppose we ask how many bits of information per area in the plane can be stored on a lattice $\beta\boldsymbol{Z}^2$ subject to a constraint $S$. This is identical to determining how many bits per area in the plane can be stored on the usual $\boldsymbol{Z}^2$-lattice using a constraint $(1/\beta)S$.

Let $\Lambda_\beta(S)$ be the number of $\beta\boldsymbol{Z}^2$-lattice points in $S$. Then, the area $A(S)$ of an open set $S$ is related to $\Lambda_\beta(S)$ by the estimate

$$A(S)/\Lambda_\beta(S) \approx \beta^2$$

where the approximation becomes equality in the limit as $A(S) \to \infty$. Thus, using Corollary V.1, if the checkerboard constraint $S$ is open, convex, and symmetric, the asymptotic number of bits that can be stored per lattice point on $\beta\boldsymbol{Z}^2$ is

$$\begin{aligned} 4\delta(S) \cdot \frac{\log_2 \Lambda_\beta(S)}{\Lambda_\beta(S)} &= 4\delta(S) \cdot \frac{\log_2 \Lambda\left(\frac{1}{\beta}S\right)}{\Lambda\left(\frac{1}{\beta}S\right)} \\ &\approx 4\delta(S) \cdot \frac{\log_2 A(S) - 2\log_2 \beta}{A(S)/\beta^2}. \end{aligned}$$

The capacity per unit area in the plane is therefore asymptotically equal to the capacity per lattice point multiplied by the number of lattice points per unit area, that is,

$$\begin{aligned} 4\delta(S) \cdot \frac{\log_2 A(S) - 2\log_2 \beta}{A(S)/\beta^2} \cdot \frac{1}{\beta^2} \\ = 4\delta(S) \cdot \frac{\log_2 A(S) - 2\log_2 \beta}{A(S)}. \end{aligned}$$

Thus, for any fixed $\beta$, in the limit as $A(S) \to \infty$, the capacity still decays at the rate

$$4\delta(S) \cdot \frac{\log_2 A(S)}{A(S)}$$

even though for any fixed $\beta < 1$, the capacity is larger than for $\beta = 1$. In summary, the asymptotic results presented are independent of the scaling of the underlying lattice, although for finite-constraint areas there may be a difference.

## Acknowledgment

The authors would like to thank Pierre Loyer for pointing out the connection between Lemma II.6 and Minkowski's classical results, Károly Böröczky for some helpful references, and Alexander Vardy for clarifying the Abstract.

## References

[1] J. J. Ashley and B. H. Marcus, "Two-dimensional low-pass filtering codes," *IEEE Trans. Commun.*, vol. 46, pp. 724–727, June 1998.
[2] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*. New York: Academic, 1982.
[3] R. J. Baxter and S. K. Tsang, "Entropy of hard hexagons," *J. Phys. A*, vol. 13, pp. 1023–1030, 1980.
[4] N. J. Calkin and H. S. Wilf, "The number of independent sets in a grid graph," *SIAM J. Discr. Math.*, vol. 11, pp. 54–60, Feb. 1998.
[5] J. W. S. Cassels, *An Introduction to the Geometry of Numbers*. Berlin, Göttingen, Heidelberg, Germany: Springer-Verlag, 1959.
[6] C. H. Dowker, "On minimum circumscribed polygons," *Bull. Amer. Math. Soc.*, vol. 50, pp. 120–122, 1944.
[7] K. Engel, "On the Fibonacci number of an $m \times n$ lattice," *Fibonacci Quart.*, vol. 28, pp. 72–78, 1990.
[8] W. H. Erxleben and M. W. Marcellin, "Error-correcting two-dimensional modulation codes," *IEEE Trans. Inform. Theory*, vol. 41, pp. 1116–1126, July 1995.
[9] T. Etzion, "Cascading methods for runlength-limited arrays," *IEEE Trans. Inform. Theory*, vol. IT-43, pp. 319–324, Jan. 1997.
[10] T. Etzion and K. G. Paterson, "Zero/positive capacities of two dimensional runlength constrained arrays," in *Proc. 2001 IEEE Intl. Symp. Information Theory*, Washington, DC, June 2001, p. 269.
[11] L. Fejes Tóth, *Regular Figures*. New York: Pergamonn, 1964.
[12] S. Forchhammer and J. Justesen, "Entropy bounds for constrained twodimensional random fields," *IEEE Trans. Inform. Theory*, vol. 45, pp. 118–127, Jan. 1999.
[13] S. Halevy and R. Roth, "Parallel constrained coding with application to two-dimensional constraints," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1009–1020, May 2002.
[14] K. A. S. Immink, P. H. Siegel, and J. K. Wolf, "Codes for digital recorders," *IEEE Trans. Inform. Theory*, vol. 44, pp. 2260–2299, Oct. 1998.
[15] G. S. Joyce, "On the hard-hexagon model and the theory of modular functions," *Philos. Trans. Roy. Soc. London Ser. A*, vol. 325, no. 1588, pp. 643–702, 1988.
[16] A. Kato and K. Zeger, "On the capacity of two-dimensional run length constrained channels," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1527–1540, July 1999.
[17] P. J. Kelly and M. L. Weiss, *Geometry and Convexity: A Study in Mathematical Methods*. New York: Wiley, 1979.
[18] M. Larsen, "The problem of the kings," *Elec. J. Combin.*, vol. 2, paper R18, 1995. Available: [Online] http://www.combinatorics.org.
[19] M. P. Richey and C. A. Tracy, "Equation of state and isothermal compressibility for the hard hexagon model in the disordered regime," *J. Phys. A*, vol. 20, no. 16, pp. L1121–Ll126, 1987.
[20] C. A. Rogers, *Packing and Covering*. Cambridge, U.K.: Cambridge Univ. Press, 1974.
[21] R. M. Roth, P. H. Siegel, and J. K. Wolf, "Efficient coding schemes for the hard-square model," *IEEE Trans. Inform. Theory*, vol. 47, pp. 1166–1176, Mar. 2001.
[22] P. H. Siegel and J. K. Wolf, "Bit stuffing bounds on the capacity of 2-dimensional constrained arrays," in *Proc. 1998 IEEE Intl. Symp. Information Theory*, Cambridge, MA, Aug. 1998, p. 323.
[23] E. Soljanin, "Writing sequences on the plane," *IEEE Trans. Inform. Theory*, vol. 48, pp. 1344–1354, June 2002.
[24] R. E. Swanson and J. K. Wolf, "A new class of two-dimensional runlength-limited recording codes," *IEEE Trans. Magn.*, vol. 28, pp. 3407–3416, Nov. 1992.
[25] R. Talyansky, T. Etzion, and R. M. Roth, "Efficient code construction for certain two-dimensional constraints," *IEEE Trans. Inform. Theory*, vol. 45, pp. 794–799, Mar. 1999.
[26] C. A. Tracy, L. Grove, and M. F. Newman, "Modular properties of the hard-hexagon model," *J. Statist. Phys.*, vol. 48, no. 3–4, pp. 477–502, 1987.
[27] B. V. Vasic, "Capacity of channels with redundant multitrack $(d, k)$ constraints: The $k \geq d$ case," *IEEE Trans. Inform. Theory*, vol. 42, pp. 1546–1548, Sept. 1996.
[28] B. V. Vasic, S. W. McLaughlin, and O. Milenkovic, "Shannon capacity of mary redundant multitrack runlength limited codes," *IEEE Trans. Inform. Theory*, vol. 44, pp. 766–774, Mar. 1998.
[29] W. Weeks and R. E. Blahut, "The capacity and coding gain of certain checkerboard codes," *IEEE Trans. Inform. Theory*, vol. 44, pp. 1193–1203, May 1998.
[30] H. S. Wilf, "The problem of the kings," *Elec. J. Combin.*, vol. 2, paper R3, 1995. Available: [Online] http://www.combinatorics.org.