Correspondence

Capacity Bounds for the Three-Dimensional $(0, 1)$ Run Length Limited Channel

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***Abstract*—The capacity $C_{0,1}^{(3)}$ of a three-dimensional $(0, 1)$ run length constrained channel is shown to satisfy $0.522501741838 \leq C_{0,1}^{(3)} \leq 0.526880847825$.**

***Index Terms*—Channel capacity, constrained codes, magnetic and optical storage.**

I. Introduction

A binary sequence satisfies a one-dimensional (d, k) run length constraint if every run of zeros has length at least d and at most k (if two ones are adjacent in the sequence we say that a run of zeros of length zero is between them). An n-dimensional binary array is said to satisfy a (d, k) run length constraint, if it satisfies the one-dimensional (d, k) run length constraint along every direction parallel to a coordinate axis. Such an array is called *valid*. The number of valid n-dimensional arrays of size $m_1 \times m_2 \times \cdots \times m_n$ is denoted by $N_{m_1, m_2, \cdots, m_n}^{(d,k)}$ and the corresponding *capacity* is defined as

$$C_{d,k}^{(n)} = \lim_{m_1, m_2, \cdots m_n \to \infty} \frac{\log_2 N_{m_1, m_2, \cdots m_n}^{(d,k)}}{m_1 m_2 \cdots m_n}.$$

By exchanging the roles of 0 and 1 it can be seen that $C_{0,1}^{(n)} = C_{1,\infty}^{(n)}$ for all $n \geq 1$. A simple proof of the existence of the two-dimensional (d, k) capacities can be found in [1], and the proof can be generalized to n dimensions.

It is known (e.g., see [2]) that the one-dimensional $(0, 1)$-constrained capacity is the logarithm of the golden ratio, i.e.,

$$C_{0,1}^{(1)} = \log_2 \frac{1 + \sqrt{5}}{2} = 0.694242 \cdots$$

and in [3] very close upper and lower bounds were given for the two-dimensional $(0, 1)$-constrained capacity. The bounds in [3] were calculated with greater precision in [4] and are slightly improved here (see Section IV for more details), now agreeing in nine decimal positions

$$0.587891161775 \leq C_{0,1}^{(2)} \leq 0.587891161868. \tag{1}$$

These bounds were also independently obtained to eight decimal positions in [5]. A lower bound of $C_{0,1}^{(2)} \geq 0.5831$ was obtained in [6] by using an implementable encoding procedure known as “bit-stuffing.” The known bounds on $C_{0,1}^{(2)}$ have played a useful role in [1] for obtaining bounds on other (d, k)-constraints in two dimensions. The three-dimensional $(0, 1)$-constrained bounds given in this correspondence can play a similar role for obtaining different three-dimensional bounds, and are also of theoretical interest. In fact, a recent tutorial paper [7] discusses an interesting connection between run length constrained capacities in more than one dimension and crossword puzzles (based on the work of Shannon from 1948). In this correspondence we consider the three-dimensional $(0, 1)$ constraint, and by extending ideas from [3] and using two new bounds, our main result is to derive (in Sections II and III) the following bounds on the three-dimensional $(0, 1)$ capacity.

Theorem 1:

$$0.522501741838 \leq C_{0,1}^{(3)} \leq 0.526880847825.$$

It is assumed henceforth in this correspondence that $d = 0$ and $k = 1$. Two valid $m_1 \times m_2$ rectangles can be put next to each other in three dimensions without violating the three-dimensional $(0, 1)$ constraint if they have no two zeros in the same positions. Define a *transfer matrix* T_{m_1, m_2} to be an $N_{m_1, m_2}^{(0,1)} \times N_{m_1, m_2}^{(0,1)}$ binary matrix, such that the rows and columns are indexed by the valid two-dimensional $m_1 \times m_2$ patterns, and an entry of T_{m_1, m_2} is 1 if and only if the corresponding two rectangles can be placed next to each other in three dimensions without violating the $(0, 1)$ constraint. Then

$$N_{m_1, m_2, m_3}^{(0,1)} = \mathbf{1}' \cdot T_{m_1, m_2}^{m_3 - 1} \mathbf{1} = \mathbf{1}' \cdot T_{m_1, m_3}^{m_2 - 1} \mathbf{1} = \mathbf{1}' \cdot T_{m_2, m_3}^{m_1 - 1} \mathbf{1} \tag{2}$$

where $\mathbf{1}$ is the all-ones column vector and prime denotes transpose. The matrix T_{m_1, m_2} meets the conditions of the Perron–Frobenius theorem [8], since it has nonnegative real elements and is irreducible (since the all-one's rectangle can be placed next to any valid rectangle without violating the $(0, 1)$ constraint). Therefore, the largest magnitude eigenvalue Λ_{m_1, m_2} of T_{m_1, m_2} is positive, real, and has multiplicity one. This implies that

$$\lim_{m_3 \to \infty} \left(N_{m_1, m_2, m_3}^{(0,1)} \right)^{1/m_3} = \Lambda_{m_1, m_2}$$

and

$$\begin{aligned} C_{0,1}^{(3)} &= \lim_{m_1, m_2, m_3 \to \infty} \frac{\log_2 N_{m_1, m_2, m_3}^{(0,1)}}{m_1 m_2 m_3} \\ &= \lim_{m_1, m_2 \to \infty} \frac{\log_2 \lim_{m_3 \to \infty} \left(N_{m_1, m_2, m_3}^{(0,1)} \right)^{1/m_3}}{m_1 m_2} \\ &= \lim_{m_1, m_2 \to \infty} \frac{\log_2 \Lambda_{m_1, m_2}}{m_1 m_2} \\ &= \lim_{m_1 \to \infty} \frac{\log_2 \lim_{m_2 \to \infty} \Lambda_{m_1, m_2}^{1/m_2}}{m_1} \\ &= \lim_{m_1 \to \infty} \frac{\log_2 \Lambda_{m_1}}{m_1} \end{aligned} \tag{3}$$

where

$$\Lambda_{m_1} = \lim_{m_2 \to \infty} \Lambda_{m_1, m_2}^{1/m_2}.$$

The quantities $\log_2 \Lambda_{m_1, m_2} / (m_1 m_2)$ and $\log_2 \Lambda_{m_1} / m_1$ can be viewed as capacities corresponding to three-dimensional arrays with two fixed sides (lengths m_1 and m_2), and one fixed side (length m_1), respectively.

Upper and lower bounds on the three-dimensional capacity can be computed directly from the inequalities (similar to the two-dimensional case, as noted in [4])

$$\frac{\log_2 \Lambda_{m_1, m_2}}{(m_1 + 1)(m_2 + 1)} \leq C_{0,1}^{(3)} \leq \frac{\log_2 \Lambda_{m_1, m_2}}{m_1 m_2}$$

but these do not yield particularly tight bounds for values of m_1 and m_2 for which the corresponding value of Λ_{m_1, m_2} could be computed by us. (For example, Table I shows that the eigenvalues Λ_{m_1, m_2} correspond to matrices with more than 40 million elements when roughly $m_1 m_2 \geq 20$.) The upper and lower capacity bounds derived in this

Manuscript received December 29, 1998. This research was supported in part by the National Science Foundation.

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Communicated by R.. M. Roth, Associate Editor for Coding Theory.
Publisher Item Identifier S 0018-9448(00)02896-0.

correspondence agree to within ± 0.002 and were computed using less than 100 Mbytes of computer memory.

II. Lower Bound on $C_{0,1}^{(3)}$

To derive a lower bound on $C_{0,1}^{(3)}$ we generalize a method of Calkin and Wilf [3]. Since T_{m_1,m_2} is a symmetric matrix, the Courant-Fischer Minimax Theorem [9, p. 394] implies that

$$\Lambda_{m_1,m_2}^p \geq \frac{\boldsymbol{x}' \cdot T_{m_1,m_2}^p \boldsymbol{x}}{\boldsymbol{x}' \cdot \boldsymbol{x}} \tag{4}$$

for any nonzero vector $\boldsymbol{x}$ and any integer $p \geq 0$. Choosing $\boldsymbol{x} = T_{m_1,m_2}^q \mathbf{1}$ for any integer $q \geq 0$, and using identity (2) gives

$$\Lambda_{m_1,m_2}^p \geq \frac{\mathbf{1}' \cdot T_{m_1,m_2}^{p+2q}\mathbf{1}}{\mathbf{1}' \cdot T_{m_1,m_2}^{2q}\mathbf{1}} = \frac{\mathbf{1}' \cdot T_{m_1,p+2q+1}^{m_2-1}\mathbf{1}}{\mathbf{1}' \cdot T_{m_1,2q+1}^{m_2-1}\mathbf{1}}. \tag{5}$$

Thus

$$\begin{aligned} 2^{pC_{0,1}^{(3)}} &= \left(\lim_{m_1,m_2\to\infty} \Lambda_{m_1,m_2}^{1/(m_1m_2)}\right)^p \\ &= \lim_{m_1\to\infty}\left(\lim_{m_2\to\infty}\Lambda_{m_1,m_2}^{p/m_2}\right)^{1/m_1} \\ &\geq \lim_{m_1\to\infty}\left(\frac{\Lambda_{m_1,p+2q+1}}{\Lambda_{m_1,2q+1}}\right)^{1/m_1} \\ &= \frac{\lim_{m_1\to\infty}\Lambda_{m_1,p+2q+1}^{1/m_1}}{\lim_{m_1\to\infty}\Lambda_{m_1,2q+1}^{1/m_1}} = \frac{\Lambda_{p+2q+1}}{\Lambda_{2q+1}} \end{aligned} \tag{6}$$

and, therefore, for any odd integer $r \geq 1$ and any integer $z > r$

$$C_{0,1}^{(3)} \geq \frac{1}{z-r}\log_2\left(\frac{\Lambda_z}{\Lambda_r}\right). \tag{7}$$

This lower bound on $C_{0,1}^{(3)}$ is analogous to a two-dimensional bound in [3], but Λ_z and Λ_r are not eigenvalues associated with transfer matrices of two-dimensional arrays here, and cannot easily be computed as in the two-dimensional case. Instead, we obtain a lower bound on Λ_z and an upper bound on Λ_r. From (5) and (6) a lower bound on Λ_z is

$$\begin{aligned} \Lambda_z &= \lim_{m_2\to\infty}\Lambda_{z,m_2}^{1/m_2} \geq \lim_{m_2\to\infty}\left(\frac{\mathbf{1}'\cdot T_{z,v}^{m_2-1}\mathbf{1}}{\mathbf{1}'\cdot T_{z,u}^{m_2-1}\mathbf{1}}\right)^{1/((v-u)m_2)} \\ &= \left(\frac{\Lambda_{z,v}}{\Lambda_{z,u}}\right)^{1/(v-u)} \end{aligned}$$

where u is an arbitrary positive odd integer, $v > u$, and $\Lambda_{z,v}$ and $\Lambda_{z,u}$ are the largest eigenvalues of the transfer matrices $T_{z,v}$ and $T_{z,u}$, respectively.

To find an upper bound on the quantity Λ_r for a given r, we apply a modified version of a method in [3]. We say that a binary matrix satisfies the $(0,1)$ *cylindrical constraint* if it satisfies the usual two-dimensional $(0,1)$ constraint after joining its leftmost column to its rightmost column (i.e., the left and right columns can be put next to each other without violating the $(0,1)$ constraint). A binary matrix satisfies the $(0,1)$ *toroidal constraint* if it satisfies the usual two-dimensional $(0,1)$ constraint after both joining its leftmost column to its rightmost column, and its top row to its bottom row.

Proposition 1: Let s be a positive integer and let T_{m_1,m_2} be the transfer matrix whose rows and columns are indexed by all $(0,1)$-constrained $m_1 \times m_2$ rectangles. Let $B_{m_1,s}$ denote the transfer matrix whose rows and columns are indexed by all cylindrically $(0,1)$-constrained $m_1 \times s$ rectangles. Then

$$\mathrm{Trace}\left[T_{m_1,m_2}^s\right] = \mathbf{1}' \cdot B_{m_1,s}^{m_2-1}\mathbf{1}.$$

Proof: $\mathrm{Trace}[T_{m_1,m_2}^s]$ is the number of $m_1 \times m_2 \times (s+1)$ valid arrays, whose first and last $m_1 \times m_2$ rectangles are the same, or equivalently the number of three-dimensional $m_1 \times m_2 \times s$ valid arrays, whose first $m_1 \times m_2$ rectangle can be put after the last one without violating the $(0,1)$ constraint. Viewing these three-dimensional arrays along their side of length m_2, they can be described as a sequence of m_2 cylindrically $(0,1)$-constrained two-dimensional rectangles of size $m_1 \times s$ (see Fig. 1), and thus the number of arrays counting in this manner is the sum of the entries in $B_{m_1,s}^{m_2-1}$. □

The proof above generalizes the two-dimensional version in [3]. Let s be a positive even integer. Then for every positive integer m_1 and m_2, the matrix T_{m_1,m_2}^s has nonnegative eigenvalues and thus any one of its eigenvalues is upper-bounded by its trace. Hence

$$\Lambda_{m_1,m_2} \leq \mathrm{Trace}\left[T_{m_1,m_2}^s\right]^{1/s} = \left(\mathbf{1}' \cdot B_{m_1,s}^{m_2-1}\mathbf{1}\right)^{1/s} \tag{8}$$

which gives the following upper bound on Λ_r:

$$\Lambda_r = \lim_{m_2\to\infty}\Lambda_{r,m_2}^{1/m_2} \leq \lim_{m_2\to\infty}\left(\mathbf{1}'\cdot B_{r,s}^{m_2-1}\mathbf{1}\right)^{\frac{1}{sm_2}} = \xi_{r,s}^{1/s} \tag{9}$$

where $\xi_{r,s}$ is the largest eigenvalue of $B_{r,s}$ (note that $B_{r,s}$ satisfies the Perron-Frobenius theorem for the same reasons as for T_{m_1,m_2} in Section I).

The lower bound on $C_{0,1}^{(3)}$ in (7) can now be written as

$$C_{0,1}^{(3)} \geq \frac{1}{z-r}\log_2\left(\frac{\left(\frac{\Lambda_{z,v}}{\Lambda_{z,u}}\right)^{1/(v-u)}}{\xi_{r,s}^{1/s}}\right), \quad \begin{array}{l} r \text{ and } u \text{ odd, } s \text{ even} \\ z > r \geq 1 \\ v > u \geq 1 \\ s \geq 2. \end{array} \tag{10}$$

To obtain the best possible lower bound, the right-hand side of (10) should be maximized over all acceptable choices of r, z, u, v, and s, subject to the numerical computability of the quantities $\Lambda_{z,v}$, $\Lambda_{z,u}$, and $\xi_{r,s}$. Table I shows the largest eigenvalues of various transfer matrices which were numerically computable. From this table, the best parameters we could find for the lower bound in (10) on the capacity were $r=3$, $z=4$, $u=5$, $v=6$, and $s=10$, yielding

$$C_{0,1}^{(3)} \geq \frac{1}{4-3}\log_2\frac{\frac{9346.35893701}{2102.73425568}}{(80481.0598379)^{1/10}} \geq 0.522501741838.$$

III. Upper Bound on $C_{0,1}^{(3)}$

Proposition 2: Let s_1 and s_2 be positive even integers and let B_{s_1,s_2}^* denote the transfer matrix whose rows and columns are indexed by all toroidally $(0,1)$-constrained $s_1 \times s_2$ rectangles. If ξ_{s_1,s_2}^* is the largest eigenvalue of B_{s_1,s_2}^*, then $C_{0,1}^{(3)} \leq 1/(s_1s_2)\log_2\xi_{s_1,s_2}^*$.

Proof: Let T_{m_1,m_2} and B_{m_1,s_1} be the same transfer matrices as defined in Section II, and let ξ_{m_1,s_1} denote the largest eigenvalue of B_{m_1,s_1}. From Proposition 1 and the argument used to obtain inequality (9) we can also conclude that

$$\Lambda_{m_1} \leq \xi_{m_1,s_1}^{1/s_1}.$$

Also, the same argument used to obtain (8) gives

$$\xi_{m_1,s_1} \leq \left(\mathrm{Trace}\left[B_{m_1,s_1}^{s_2}\right]\right)^{1/s_2} = \left(\mathbf{1},(B_{s_1,s_2}^*)^{m_1-1}\mathbf{1}\right)^{1/s_2}$$

and thus

$$\Lambda_{m_1}^{1/m_1} \leq \xi_{m_1,s_1}^{1/(m_1s_1)} \leq \left(\mathbf{1},(B_{s_1,s_2}^*)^{m_1-1}\mathbf{1}\right)^{1/(m_1s_1s_2)}.$$

This uses the fact that B_{s_1,s_2}^* satisfies the Perron-Frobenius theorem (for the same reasons as for T_{m_1,m_2} in Section I). Since

$$C_{0,1}^{(3)} = \lim_{m_1\to\infty}\log_2\Lambda_{m_1}^{1/m_1}$$

we have

$$2^{C_{0,1}^{(3)}} = \lim_{m_1\to\infty}\Lambda_{m_1}^{1/m_1} \leq \left(\xi_{s_1,s_2}^*\right)^{1/(s_1s_2)}.$$ □

Proposition 2 generalizes an upper bound in [3] and is illustrated in Fig. 2. Note that $B_{2,s_2} = B_{2,s_2}^*$ and thus $\xi_{2,s_2} = \xi_{2,s_2}^*$. The best parameters we were able to find (from Table I) were $s_1 = 4$ and $s_2 = 6$, and the resulting eigenvalue gave the following upper bound:

$$C_{0,1}^{(3)} \leq \frac{1}{24}\log_2 6405.69924332 \leq 0.526880847825.$$

TABLE I
THE LARGEST EIGENVALUES OF THE TRANSFER MATRICES $T_{a,b}$, $B_{a,b}$, AND $B^*_{a,b}$ ARE $\Lambda_{a,b}$, $\xi_{a,b}$, AND $\xi^*_{a,b}$, RESPECTIVELY. THE VALUES FOR $B_{a,b}$ ARE ONLY GIVEN WHEN b IS EVEN, AND FOR $B^*_{a,b}$ WHEN BOTH a AND b ARE EVEN. EIGENVALUE ENTRIES IN THE TABLE WITH AN "*" NEXT TO THEM INDICATE THAT THEY WERE COMPUTED USING THE POWER METHOD INSTEAD OF BY DIRECT COMPUTATION (SEE SECTION IV). THE EIGENVALUES $\Lambda_{a,b}$ and $\xi_{a,b}$ ARE SYMMETRIC IN THEIR INDICES

a	b	$\Lambda_{a,b}$	rows of $T_{a,b}$	$\xi_{a,b}$	rows of $B_{a,b}$	$\xi^*_{a,b}$	rows of $B^*_{a,b}$
1	1	1.61803398875	2				
	2	2.41421356237	3	2.41421356237	3		
	3	3.63138126040	5				
	4	5.45770539597	8	5.15632517466	7		
	5	8.20325919376	13				
	6	12.3298822153	21	11.5517095660	18		
	7	18.5324073775	34				
	8	27.8550990963	55	26.0579860919	47		
	9	41.8675533183	89				
	10	62.9289457252	144	58.8519350815	123		
	11	94.5852312050	233				
	12	142.166150393	377	132.947794048	322		
	13	213.682559741	610				
	14	321.175161677	987	300.345852027	843		
	15	482.741710897	1597				
	16	725.584002895*	2584	678.525669346	2207		
	17	1090.58764423*	4181				
	18	1639.20566742*	6765	1532.89283597*	5778		
	19	2463.80493521*	10946				
	20	3703.21728345*	17711	3463.03987027*	15127		
	21	5566.11363689*	28657				
	22	8366.13642876*	46368	7823.53857819*	39603		
	23	12574.7053170*	75025				
	24	18900.3867144*	121393	17674.5747630*	103682		
2	2	5.15632517466	7	5.15632517466	7	5.15632517466	7
	3	11.1103016575	17				
	4	23.9250625386	41	21.9287654025	35	21.9287654025	35
	5	51.5229210280	99				
	6	110.954925971	239	100.236549238	199	100.236549239	199
	7	238.942175857	577				
	8	514.563569622	1393	463.203410887	1155	463.203410887	1155
	9	1108.11608218*	3363				
	10	2386.33538059*	8119	2146.04060032*	6727	2146.04060032*	6727
	11	5138.98917320*	19601				
	12	11066.8474924*	47312	9949.63685703*	39203	9949.63685703*	39203
3	3	34.4037405361	63				
	4	106.439377528	227	94.2548937790	181		
	5	329.331697608	827				
	6	1018.97101980*	2999	884.498791440	2309		
	7	3152.75734322*	10897				
	8	9754.81971205*	39561	8421.60680806*	30277		
	9	30181.9963196*	143677				
	10	93384.9044989*	521721	80481.0598378*	398857		
4	4	473.069084944	1234	404.943621498	933	355.525781764	743
	5	2102.73425567*	6743				
	6	9346.35893702*	36787	7799.87080772*	26660	6405.69924332*	18995

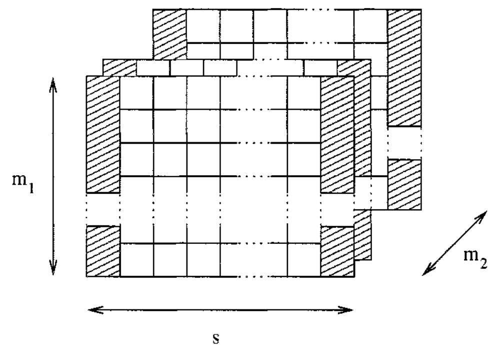

Fig. 1. Cylindrically $(0, 1)$-constrained $m_1 \times s$ rectangles used to build cylindric $m_1 \times m_2 \times s$ arrays.

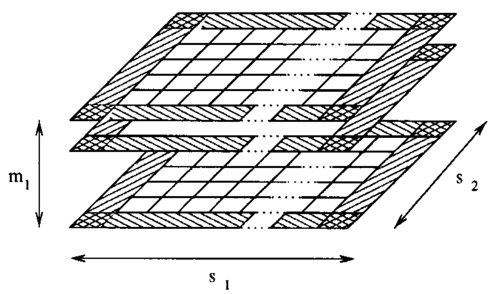

Fig. 2. Toroidally $(0, 1)$-constrained $s_1 \times s_2$ rectangles used to build doubly cylindric $m_1 \times s_1 \times s_2$ arrays.

IV. Remark

Direct computation of eigenvalues using standard linear algebra algorithms generally requires the storage of an entire matrix. This severely restricts the matrix sizes allowable, due to memory constraints on computers. By exploiting the fact that our matrices are all binary, symmetric, and easily computable, we were able to obtain the largest eigenvalues of much larger matrices. Specifically, the eigenvalues used to obtain the capacity bounds in Theorem 1 were computed using the following result.

Lemma 1 ([10, p. 493]): Let A be an $n \times n$ matrix with nonnegative entries only. Then for any n-dimensional positive vector $\boldsymbol{x}$ we have

$$\min_{1 \le i \le n} \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j \le \rho(A) \le \max_{1 \le i \le n} \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j$$

and

$$\min_{1 \le j \le n} x_j \sum_{i=1}^{n} \frac{a_{ij}}{x_i} \le \rho(A) \le \max_{1 \le j \le n} x_j \sum_{i=1}^{n} \frac{a_{ij}}{x_i}$$

where $\rho(A)$ denotes the spectral radius of the matrix A.

The convergence rate of the power method depends on the relative size of the largest and second largest eigenvalues, but the second largest eigenvalues are generally unknown to us. Hence, we iterated the eigenvalue computation until the eigenvalues appeared to stabilize in the 14th significant decimal place (computing $\Lambda_{4,5}$, $\Lambda_{4,6}$, $\xi_{3,10}$, and $\xi^*_{4,6}$). The resulting eigenvector estimates were used as the values of $\boldsymbol{x}$ in Lemma 1 to obtain *exact* upper and lower bounds on the largest eigenvalues.

Similarly, we obtained the upper bound in (1) with the power method (computing $\Lambda_{1,21}$, $\Lambda_{1,23}$, and $\xi_{1,24}$). Originally these bounds were computed in [3] as

$$0.587891161 \le C_{0,1}^{(2)} \le 0.588339078$$

(computing $\Lambda_{1,13}$, $\Lambda_{1,15}$, and $\xi_{1,6}$) and were later improved in [4] (computing $\Lambda_{1,13}$, $\Lambda_{1,14}$, and $\xi_{1,14}$) to

$$0.587891161775 \le C_{0,1}^{(2)} \le 0.587891494943.$$

The lower bound in (1) is from [4].

We expect the bounds in (10) and in Proposition 2 to improve in the future as increased computational speed and memory expand more of Table I.

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