Randomly Chosen Index Assignments Are Asymptotically Bad for Uniform Sources

András Méhes and Kenneth Zeger, *Senior Member, IEEE*

***Abstract*— It is known that among all redundancy-free codes (or index assignments), the natural binary code minimizes the mean-squared error (MSE) of the uniform source and uniform quantizer on a binary symmetric channel. We derive a code which maximizes the MSE and demonstrate that the code is linear and its distortion is asymptotically equivalent, as the blocklength grows, to the expected distortion of an index assignment chosen uniformly at random.**

***Index Terms*—Index assignment, noisy channel vector quantization.**

I. Introduction

An index assignment is a mapping of source code symbols to channel code symbols. The usual goal of index assignment design for noisy channel vector quantizers is to minimize the end-to-end mean-squared error (MSE) over all possible index assignments. The MSE is computed with respect to the statistics of both the source and channel. Previous work has examined the theoretical and practical aspects of index assignment in noisy channel vector quantizer systems. In particular, it is known that the performance of such a system can be significantly affected by the choice of index assignment.

The problem of algorithmically finding good index assignments has been previously studied in [1]–[6], and analytic formulas have been found for binary symmetric channels and certain sources [7]–[12]. The optimality of the natural binary code was conjectured in [8] and proved in [10] for uniform scalar quantization of a uniform source and later extended to binary lattice vector quantizers with equiprobable quantization points in [11].

In this paper, we derive an index assignment which *maximizes* the MSE for a uniform scalar source and show that the worst case performance thus obtained is asymptotically equivalent to the expected performance of an index assignment chosen uniformly at random. This indicates that the majority of index assignments are asymptotically bad. Also, this result analytically reveals the entire range of possible performances achievable by different index assignments.

The overall MSE of a quantizer optimized for a noiseless channel can be decomposed into a "source distortion" due to quantization and a "channel distortion" due to channel noise [13]. The source component is a result of representing the source with a finite number of quantization points and thus is independent of the index assignment. The channel component, on the other hand, results from confusing the indices of quantization points because of channel errors. Hence, we focus on the channel distortion, when evaluating index assignments. With this in mind, the index assignment problem can be reformulated as a discrete problem with no direct reference to quantization. For n-bit uniform scalar quantization of a uniform source, the quantization points are scaled and translated versions of $0, \cdots, 2^n - 1$. The usual index assignment problem is to assign indices to quantization points to minimize the mean squared distance between pairs of these points, with respect to their prior probabilities and the channel transition probabilities. In this paper, however, we maximize the MSE.

Manuscript received June 1, 1997; revised July 12, 1998. This work was supported in part by the National Science Foundation.

The authors are with the Department of Electrical and Computer Engineering, University of California, San Diego, CA 92093-0407 USA (e-mail: andras@ucsd.edu; zeger@ucsd.edu).

Communicated by N. Merhav, Associate Editor for Source Coding.
Publisher Item Identifier S 0018-9448(99)01413-3.

The paper is organized as follows. Section II gives notation and definitions. In Section III, we derive a distortion-maximizing index assignment (the worst code) for uniform scalar quantization of a uniform source (Theorem 1) and compare the performances of the best, worst, and randomly chosen index assignments (Corollary 1). A counterexample in Section IV shows that the MSE-maximizing property of the worst code does not extend to arbitrary binary lattice vector quantizers (Corollary 3), even though it is known that the MSE-minimizing property of the natural binary code does extend to binary lattice vector quantizers. We establish, however, that among all affine index assignments, the worst code does maximize the MSE of arbitrary binary lattice vector quantizers (Corollary 2).

II. Preliminaries

For any positive integer n, let Z_2^n denote the field of n-bit binary words, where arithmetic is performed modulo 2. Every integer $i \in \mathcal{S} = \{0, \cdots, 2^n - 1\}$ has a unique binary representation $i = \sum_{l=0}^{n-1} 2^l i_l$, where $i_l \in \{0, 1\}$. We denote by $\mathbf{i} \in Z_2^n$ the binary n-tuple (row vector) corresponding to i, i.e.,

$$\mathbf{i} = [i_{n-1}, i_{n-2}, \cdots, i_1, i_0].$$

The transpose of $\mathbf{i} \in Z_2^n$ is denoted by $\mathbf{i}^T$. For $\mathbf{i}, \mathbf{j} \in Z_2^n$, $\mathbf{i}^T\mathbf{j}$ is a binary matrix, while $\mathbf{i}\mathbf{j}^T = \sum_{l=0}^{n-1} i_l j_l \in \{0, 1\}$ is the binary inner product of the two vectors. We denote by $\mathbf{e}^{(m)}$ the binary vector corresponding to 2^m, i.e., $e_l^{(m)} = I_{\{m=l\}}$, where I is the indicator function. The all-zero vector is denoted by $\mathbf{0}$ and the all-one vector by $\mathbf{1}$.

Definition 1: An *index assignment* is a mapping $\pi : Z_2^n \to Z_2^n$ which is a bijection. An index assignment is a permutation of Z_2^n, and thus there are $(2^n)!$ different index assignments.

An *affine index assignment* $\pi : Z_2^n \to Z_2^n$ is an index assignment of the form

$$\pi(\mathbf{i}) = \mathbf{i}\mathsf{G} + \mathbf{t}, \quad \pi^{-1}(\mathbf{i}) = (\mathbf{i} + \mathbf{t})\mathsf{G}^{-1}$$

where G is a binary nonsingular $n \times n$ *generator matrix*, $\mathbf{t}$ is an n-dimensional binary *translation vector*, and the arithmetic is performed in Z_2^n. If $\mathbf{t} = \mathbf{0}$, then π is called *linear*.

The family of affine index assignments is attractive due to its low-implementation complexity and was first systematically studied in [12] and [14]–[16]. An unstructured index assignment requires a table of size $O(n2^n)$ bits to implement, whereas affine assignments can be described by $O(n^2)$ bits. Many useful index assignments are known to be affine, including the natural binary code, folded binary code, gray code, and two's complement code [12].

Definition 2; The *natural binary code* π_N is the identity index assignment $\pi_N(\mathbf{i}) = \mathbf{i}$. It is a linear index assignment with generator matrix $\mathsf{G}_N = \mathsf{I}$, the identity matrix.

TABLE I
A 4-b EXAMPLE OF THE NATURAL BINARY CODE AND THE WORST CODE.

i	$\pi_N(\mathbf{i})$		$\pi_W(\mathbf{i})$	
0	0	0000	0	0000
1	1	0001	9	1001
2	2	0010	10	1010
3	3	0011	3	0011
4	4	0100	12	1100
5	5	0101	5	0101
6	6	0110	6	0110
7	7	0111	15	1111
8	8	1000	7	0111
9	9	1001	14	1110
10	10	1010	13	1101
11	11	1011	4	0100
12	12	1100	11	1011
13	13	1101	2	0010
14	14	1110	1	0001
15	15	1111	8	1000

We define the *worst code* π_W to be the linear index assignment with generator matrix

$$\mathsf{G}_W = \begin{bmatrix} n & 1 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

where the "n" in the top-left component of G_W is taken modulo 2. The inverse of the generator matrix is

$$\mathsf{G}_W^{-1} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}.$$

Table I gives an explicit listing (in both decimal and binary) of these two index assignments for $n = 4$.

Let the channel transition probabilities of a binary symmetric channel be denoted by (for $\epsilon < 1/2$)

$$p(a \mid b) = \begin{cases} 1-\epsilon, & \text{if } a = b \\ \epsilon, & \text{if } a \neq b \end{cases} \quad a, b \in \{0, 1\}.$$

Definition 3: The *Hamming weight* of a binary n-tuple $\mathbf{a} \in Z_2^n$ is the number of its nonzero components

$$w(\mathbf{a}) \triangleq \sum_{l=0}^{n-1} I_{\{a_l \neq 0\}}.$$

The *transition probabilities* for binary n-tuples on a binary symmetric channel are

$$P(\mathbf{a} \mid \mathbf{b}) = \prod_{l=0}^{n-1} p(a_l \mid b_l) = \epsilon^{w(\mathbf{a}+\mathbf{b})}(1-\epsilon)^{n-w(\mathbf{a}+\mathbf{b})}, \quad \mathbf{a}, \mathbf{b} \in Z_2^n.$$

We denote the probability that an error pattern $\mathbf{a} \in Z_2^n$ occurs on a binary symmetric channel by

$$\rho_{\mathbf{a}} \triangleq P(\mathbf{b} + \mathbf{a} \mid \mathbf{b}) = \epsilon^{w(\mathbf{a})}(1-\epsilon)^{n-w(\mathbf{a})}, \quad \mathbf{b} \in Z_2^n. \tag{1}$$

Definition 4: Let π be an index assignment and suppose an element i is chosen uniformly at random from the set $\mathcal{S} = \{0, \cdots, 2^n - 1\}$, where the binary n-tuple $\pi(\mathbf{i})$ is transmitted over a binary symmetric channel with error probability ϵ. The end-to-end MSE is defined as

$$D \triangleq 2^{-n} \sum_{i \in \mathcal{S}} \sum_{j \in \mathcal{S}} (i-j)^2 \rho_{\pi(\mathbf{i})+\pi(\mathbf{j})}. \tag{2}$$

It may be assumed without loss of generality that $\pi(\mathbf{0}) = \mathbf{0}$, which for an affine index assignment $\pi(\mathbf{i}) = \mathbf{i}\mathsf{G} + \mathbf{t}$ is equivalent to setting $\mathbf{t} = \mathbf{0}$. Thus, we omit the translation vector $\mathbf{t}$ in what follows.

Definition 5: For each $\mathbf{i}, \mathbf{j} \in Z_2^n$, let $h_{\mathbf{i},\mathbf{j}} = (-1)^{\mathbf{i}\mathbf{j}^T}$. The *Hadamard transform* $\hat{f} : Z_2^n \to R$ of a mapping $f : Z_2^n \to R$ is defined by

$$\hat{f}(\mathbf{j}) = \sum_{\mathbf{i} \in Z_2^n} f(\mathbf{i}) h_{\mathbf{i},\mathbf{j}}$$

and the inverse transform is given by

$$f(\mathbf{i}) = 2^{-n} \sum_{\mathbf{j} \in Z_2^n} \hat{f}(\mathbf{j}) h_{\mathbf{j}\mathbf{i}}.$$

The Hadamard transform provides a tool for analyzing the mean-squared distortion [5], [12], [16]–[19]. The following properties of Hadamard transforms are useful. For any $\mathbf{i}, \mathbf{j}, \mathbf{a}, \mathbf{b} \in Z_2^n$:

1)
$$h_{\mathbf{i},\mathbf{j}} = h_{\mathbf{j},\mathbf{i}}$$

2)
$$h_{\mathbf{i},\mathbf{a}+\mathbf{b}} = h_{\mathbf{i},\mathbf{a}} h_{\mathbf{i},\mathbf{b}}$$

3)
$$\sum_{\mathbf{i} \in Z_2^n} h_{\mathbf{i},\mathbf{j}} = \begin{cases} 2^n, & \text{if } \mathbf{j} = \mathbf{0} \\ 0, & \text{otherwise} \end{cases}$$

4)
$$\sum_{\mathbf{i} \in Z_2^n} i_m h_{\mathbf{i},\mathbf{j}} = \begin{cases} 2^{n-1}, & \text{if } \mathbf{j} = \mathbf{0} \\ -2^{n-1}, & \text{if } \mathbf{j} = \mathbf{e}^{(m)}, \quad m \in \{0, 1, \cdots, n-1\} \\ 0, & \text{otherwise.} \end{cases}$$

The first two properties are straightforward. Property 3) follows from the fact that exactly half of the binary vectors in Z_2^n are orthogonal to any fixed nonzero vector $\mathbf{j} \in Z_2^n$. To see Property 4), let $\mathbf{i}'$, $\mathbf{j}' \in Z_2^{n-1}$, respectively, denote the binary vectors $\mathbf{i}, \mathbf{j} \in Z_2^n$, but with the mth component removed. Then, we can rewrite Property 4) as

$$\begin{aligned} &\sum_{\mathbf{i}' \in Z_2^{n-1}} \sum_{i_m \in \{0,1\}} i_m (-1)^{i_m j_m} h_{\mathbf{i}',\mathbf{j}'} \\ &= (-1)^{j_m} \sum_{\mathbf{i}' \in Z_2^{n-1}} h_{\mathbf{i}',\mathbf{j}'} = (-1)^{j_m} 2^{n-1} I_{\{\mathbf{j}'=\mathbf{0}\}} \end{aligned}$$

where the last equality follows from Property 3) for Z_2^{n-1}.

Lemma 1: The Hadamard transform $\hat{\rho}$ of the error pattern distribution ρ is

$$\hat{\rho}_{\mathbf{j}} = (1-2\epsilon)^{w(\mathbf{j})}.$$

Proof:

$$\begin{aligned}\hat{\rho}_{\mathbf{j}} &= \sum_{\mathbf{i}\in Z_2^n} \epsilon^{w(\mathbf{i})}(1-\epsilon)^{n-w(\mathbf{i})} h_{\mathbf{i},\mathbf{j}} \\ &= \sum_{i_0\in\{0,1\}}\sum_{i_1\in\{0,1\}}\cdots\sum_{i_{n-1}\in\{0,1\}}\left(\prod_{l=0}^{n-1}\epsilon^{i_l}(1-\epsilon)^{1-i_l}(-1)^{i_l j_l}\right) \\ &= \prod_{l=0}^{n-1}\sum_{i_l\in\{0,1\}}\epsilon^{i_l}(1-\epsilon)^{1-i_l}(-1)^{i_l j_l} = \prod_{l=0}^{n-1}(1-\epsilon+\epsilon(-1)^{j_l}) \\ &= (1-2\epsilon)^{w(\mathbf{j})}.\end{aligned}$$

□

III. Construction of the Worst Code

The following lemma gives an expression for the distortion D in the Hadamard transform domain. Variants of this result were used to show the optimality of the natural binary code in [10] and [11]. The lemma is useful for identifying a "worst" code.

Lemma 2: Let $\eta(\mathbf{i}) = \pi^{-1}(\mathbf{i})$ for all $\mathbf{i}\in Z_2^n$. Then the distortion in the Hadamard transform domain is

$$D = 2\sum_{\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}} [2^{-n}\hat{\eta}(\mathbf{a})]^2\big(1-(1-2\epsilon)^{w(\mathbf{a})}\big).$$

Proof: Rewriting (2) using η yields

$$\begin{aligned}D &= 2^{-n}\sum_{\mathbf{i}\in Z_2^n}\sum_{\mathbf{j}\in Z_2^n}(\eta(\mathbf{i})-\eta(\mathbf{j}))^2\rho_{\mathbf{i}+\mathbf{j}} \\ &= 2^{-n}\sum_{\mathbf{i}\in Z_2^n}\sum_{\mathbf{j}\in Z_2^n}\left[2^{-n}\sum_{\mathbf{a}\in Z_2^n}\hat{\eta}(\mathbf{a})(h_{\mathbf{a},\mathbf{i}}-h_{\mathbf{a},\mathbf{j}})\right]^2\rho_{\mathbf{i}+\mathbf{j}} \\ &= 8^{-n}\sum_{\mathbf{i}\in Z_2^n}\sum_{\mathbf{j}\in Z_2^n}\rho_{\mathbf{i}+\mathbf{j}}\sum_{\mathbf{a}\in Z_2^n}\sum_{\mathbf{b}\in Z_2^n}\hat{\eta}(\mathbf{a})\hat{\eta}(\mathbf{b}) \\ &\quad\times(h_{\mathbf{a}+\mathbf{b},\mathbf{i}}-h_{\mathbf{a},\mathbf{i}}h_{\mathbf{b},\mathbf{j}}-h_{\mathbf{a},\mathbf{j}}h_{\mathbf{b},\mathbf{i}}+h_{\mathbf{a}+\mathbf{b},\mathbf{j}}) \\ &= 8^{-n}\sum_{\mathbf{a}\in Z_2^n}\sum_{\mathbf{b}\in Z_2^n}\hat{\eta}(\mathbf{a})\hat{\eta}(\mathbf{b})\sum_{\mathbf{i}\in Z_2^n}h_{\mathbf{a}+\mathbf{b},\mathbf{i}} \\ &\quad\times\sum_{\mathbf{j}\in Z_2^n}\rho_{\mathbf{i}+\mathbf{j}}(h_{\mathbf{0},\mathbf{i}+\mathbf{j}}-h_{\mathbf{a},\mathbf{i}+\mathbf{j}}-h_{\mathbf{b},\mathbf{i}+\mathbf{j}}+h_{\mathbf{a}+\mathbf{b},\mathbf{i}+\mathbf{j}}) \\ &= 4^{-n}\sum_{\mathbf{a}\in Z_2^n}\sum_{\mathbf{b}\in Z_2^n}\hat{\eta}(\mathbf{a})\hat{\eta}(\mathbf{b})\left(2^{-n}\sum_{\mathbf{i}\in Z_2^n}h_{\mathbf{a}+\mathbf{b},\mathbf{i}}\right) \\ &\quad\times\left(\sum_{\mathbf{c}\in Z_2^n}\rho_{\mathbf{c}}(h_{\mathbf{0},\mathbf{c}}-h_{\mathbf{a},\mathbf{c}}-h_{\mathbf{b},\mathbf{c}}+h_{\mathbf{a}+\mathbf{b},\mathbf{c}})\right) \\ &= 4^{-n}\sum_{\mathbf{a}\in Z_2^n}\sum_{\mathbf{b}\in Z_2^n}\hat{\eta}(\mathbf{a})\hat{\eta}(\mathbf{b})I_{\{\mathbf{a}=\mathbf{b}\}}(\hat{\rho}_{\mathbf{0}}-\hat{\rho}_{\mathbf{a}}-\hat{\rho}_{\mathbf{b}}+\hat{\rho}_{\mathbf{a}+\mathbf{b}}) \\ &= 2\sum_{\mathbf{a}\in Z_2^n}\left[2^{-n}\hat{\eta}(\mathbf{a})\right]^2(\hat{\rho}_{\mathbf{0}}-\hat{\rho}_{\mathbf{a}}) \\ &= 2\sum_{\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}}\left[2^{-n}\hat{\eta}(\mathbf{a})\right]^2\big(1-(1-2\epsilon)^{w(\mathbf{a})}\big).\end{aligned}$$

□

The following bounds on D follow from Lemma 2 using $1 \le w(\mathbf{a}) \le n$ for $\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}$:

$$\begin{aligned}4\epsilon&\sum_{\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}}[2^{-n}\hat{\eta}(\mathbf{a})]^2 \\ &\le D \le 2(1-(1-2\epsilon)^n)\sum_{\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}}[2^{-n}\hat{\eta}(\mathbf{a})]^2. \qquad (3)\end{aligned}$$

The lower bound was established in [10] and can be achieved with equality if $\hat{\eta}(\mathbf{a}) = 0$ for every $\mathbf{a}\in Z_2^n$ with Hamming weight $w(\mathbf{a}) > 1$. For example, the natural binary code satisfies this requirement [10], [11]. To achieve the upper bound with equality, we must have $\hat{\eta}(\mathbf{a}) = 0$ for every $\mathbf{a}\in Z_2^n$ such that $w(\mathbf{a}) < n$, i.e., $\hat{\eta}(\mathbf{1})$ must be the only nonzero Hadamard transform component. Note, however, that for any index assignment π

$$\begin{aligned}\sum_{\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}}[2^{-n}\hat{\eta}(\mathbf{a})]^2 &= \left(\sum_{\mathbf{a}\in Z_2^n}\left[2^{-n}\sum_{\mathbf{i}\in Z_2^n}\eta(\mathbf{i})h_{\mathbf{i},\mathbf{a}}\right]^2\right) \\ &\quad - \left[2^{-n}\sum_{\mathbf{i}\in Z_2^n}\eta(\mathbf{i})h_{\mathbf{i},\mathbf{0}}\right]^2 \\ &= 2^{-n}\sum_{i\in\mathcal{S}}\sum_{j\in\mathcal{S}} ij\left(2^{-n}\sum_{\mathbf{a}\in Z_2^n}h_{\pi(i)+\pi(j),\mathbf{a}}\right) \\ &\quad - \left[2^{-n}\sum_{i\in\mathcal{S}} i\right]^2 \\ &= 2^{-n}\sum_{i\in\mathcal{S}} i^2 - \left[2^{-n}\sum_{i\in\mathcal{S}} i\right]^2 \qquad (4)\\ &= \frac{4^n-1}{12} \qquad (5)\\ &= \sigma_{\mathcal{S}}^2,\end{aligned}$$

the variance of a random variable chosen uniformly at random from $\mathcal{S}$. On the other hand, for any π

$$\begin{aligned}[2^{-n}\hat{\eta}(\mathbf{1})]^2 &= \left[2^{-n}\sum_{\mathbf{i}\in Z_2^n}\eta(\mathbf{i})h_{\mathbf{i},\mathbf{1}}\right]^2 \\ &= \left[2^{-n}\sum_{i\in\mathcal{S}} i(-1)^{\pi(i)\mathbf{1}^T}\right]^2 \\ &\le \left[2^{-n}\left(\sum_{i=2^{n-1}}^{2^n-1} i - \sum_{i=0}^{2^{n-1}-1} i\right)\right]^2 \\ &= [2^{-n}4^{n-1}]^2 \\ &= 4^{n-2} \qquad (6)\\ &< \frac{4^n-1}{12},\end{aligned}$$

for every $n > 1$. Thus, the upper bound given in (3) is not achievable when $n > 1$. The next tightest upper bound from Lemma 2 is obtained using $w(\mathbf{a}) \le n-1$ for all $\mathbf{a}\ne\mathbf{1}$, i.e.,

$$\begin{aligned}D &\le 2(1-(1-2\epsilon)^n)\max_{\eta_0}[2^{-n}\hat{\eta}_0(\mathbf{1})]^2 + 2(1-(1-2\epsilon)^{n-1}) \\ &\quad\times\left(\sum_{\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}}[2^{-n}\hat{\eta}(\mathbf{a})]^2 - \max_{\eta_0}[2^{-n}\hat{\eta}_0(\mathbf{1})]^2\right). \qquad (7)\end{aligned}$$

Indeed, the worst code achieves (7). To prove this, consider the Hadamard transform components $\hat{\eta}(\mathbf{a})$ of an arbitrary linear index assignment $\pi(\mathbf{i}) = \mathbf{i}\mathsf{G}$, and for any $\mathbf{a} \in Z_2^n \backslash \{\mathbf{0}\}$

$$\begin{aligned}\hat{\eta}(\mathbf{a}) &= \sum_{\mathbf{i}\in Z_2^n} \eta(\mathbf{i}) h_{\mathbf{i},\mathbf{a}} \\ &= \sum_{i\in\mathcal{S}} i h_{\pi(\mathbf{i}),\mathbf{a}} \\ &= \sum_{i\in\mathcal{S}} \left(\sum_{l=0}^{n-1} 2^l i_l\right) h_{\mathbf{i}G,\mathbf{a}} \\ &= \sum_{l=0}^{n-1} 2^l \left(\sum_{\mathbf{i}\in Z_2^n} i_l h_{\mathbf{i},\mathbf{a}\mathsf{G}^T}\right) \\ &= -2^{n-1} \sum_{l=0}^{n-1} 2^l I_{\{\mathbf{a}\mathsf{G}^T = \mathbf{e}^{(l)}\}} \end{aligned} \tag{8}$$

where (8) follows from Property 4) since $\mathbf{a}\mathsf{G}^T \neq \mathbf{0}$ for $\mathbf{a} \neq \mathbf{0}$ by the nonsingularity of G. Therefore, the only nonzero Hadamard transform components are those corresponding to $\mathbf{a} = \mathbf{e}^{(l)}(\mathsf{G}^T)^{-1}$, for $l = 0, \cdots, n-1$. Thus, to achieve the lower bound given in (3), every row of $(\mathsf{G}^T)^{-1}$ must have Hamming weight 1, as with the natural binary code.

Similarly, the upper bound given in (7) can be achieved by setting $\mathbf{e}^{(n-1)}(\mathsf{G}^T)^{-1} = \mathbf{1}$ (i.e., the first row of the inverse of the transposed generator matrix must be all ones) and choosing the remaining $n-1$ rows of $(\mathsf{G}^T)^{-1}$ to have Hamming weight $n-1$. An example is G_W^{-1}, given in the definition of the worst code. Note that the all-one vector has to be the first row of $(\mathsf{G}^T)^{-1}$ to ensure the maximization of

$$[2^{-n}\hat{\eta}(\mathbf{1})]^2 = [2^{-n}(-2^{n-1}2^{n-1})]^2 = 4^{n-2}.$$

Thus, combining the lower bound from (3) and the upper bound from (7) and using (5) and (6) to eliminate the remaining Hadamard transforms from the expressions, we obtain the following theorem.

Theorem 1: Suppose an integer $\mathbf{i}$ is chosen uniformly at random from the set $\mathcal{S} = \{0, \cdots, 2^n - 1\}$ and the n-bit word $\pi(\mathbf{i})$ is transmitted over a binary symmetric channel with bit error probability $\epsilon \in [0, 1/2]$, using an index assignment π. Then the resulting MSE D satisfies

$$\epsilon \frac{4^n - 1}{3} \leq D \leq \epsilon(1-2\epsilon)^{n-1}4^{n-1} + (1 - (1-2\epsilon)^{n-1})\frac{4^n - 1}{6}$$

where the lower bound is achieved by the natural binary code and the upper bound by the worst code.

Let us denote the distortion of the natural binary code and the worst code, respectively, by $D_{\min} = \epsilon\frac{4^n-1}{3}$ and $D_{\max} = \epsilon(1-2\epsilon)^{n-1}4^{n-1} + (1-(1-2\epsilon)^{n-1})\frac{4^n-1}{6}$. If an index assignment is chosen uniformly at random, then the average distortion is

$$D_{\text{ave}} \triangleq \frac{1}{(2^n)!} \sum_{\pi} 2^{-n} \sum_{i\in\mathcal{S}} \sum_{j\in\mathcal{S}} (i-j)^2 \rho_{\pi(\mathbf{i}) + \pi(\mathbf{j})}. \tag{9}$$

Since $\rho_{\mathbf{0}} = (1-\epsilon)^n$, and

$$\begin{aligned} &\frac{1}{(2^n)!} \sum_{\pi} \rho_{\pi(\mathbf{i})+\pi(\mathbf{j})} \\ &= \sum_{\mathbf{a}\in Z_2^n} \sum_{\mathbf{b}\in Z_2^n} \rho_{\mathbf{a}+\mathbf{b}} \left(\frac{1}{(2^n)!} \sum_{\pi} I_{\{\pi(\mathbf{i})=\mathbf{a}, \pi(\mathbf{j})=\mathbf{b}\}}\right) \\ &= \sum_{\mathbf{a}\in Z_2^n} \sum_{\mathbf{b}\in Z_2^n} \rho_{\mathbf{a}+\mathbf{b}} \Bigg(I_{\{\mathbf{i}=\mathbf{j}, \mathbf{a}=\mathbf{b}\}} \frac{(2^n-1)!}{(2^n)!} \\ &\quad + I_{\{\mathbf{i}\neq\mathbf{j}, \mathbf{a}\neq\mathbf{b}\}} \frac{(2^n-2)!}{(2^n)!}\Bigg) \\ &= I_{\{\mathbf{i}=\mathbf{j}\}}\rho_{\mathbf{0}} + I_{\{\mathbf{i}\neq\mathbf{j}\}} \frac{1-\rho_{\mathbf{0}}}{2^n - 1} \end{aligned}$$

this gives

$$\begin{aligned} D_{\text{ave}} &= \frac{1-(1-\epsilon)^n}{2^n(2^n-1)} \sum_{i\in\mathcal{S}} \sum_{j\in\mathcal{S}} (i-j)^2 \\ &= (1-(1-\epsilon)^n)\frac{4^n + 2^n}{6}. \end{aligned}$$

The values of $D_{\min}$ and D_{ave} were apparently first reported in [7].

Clearly, the inequalities $D_{\min} \leq D_{\text{ave}} \leq D_{\max}$ hold for every $\epsilon \in [0, 1/2]$ and $n \geq 1$. It is interesting to examine the asymptotic behavior of the minimum, maximum, and average distortions, both as the blocklength n grows and as the channel error probability ϵ decreases. The partial derivatives of $\frac{D_{\max}}{D_{\min}}$, $\frac{D_{\text{ave}}}{D_{\min}}$, and $\frac{D_{\max}}{D_{\text{ave}}}$ with respect to ϵ are all strictly negative for all $\epsilon \in (0, 1/2)$ and for all $n > 1$. Hence, the largest performance gain of a best index assignment over a worst index assignment or over an average index assignment occurs in the limit as $\epsilon \to 0$. Asymptotically as $\epsilon \to 0$, for a fixed blocklength n, these gains are given by

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{D_{\max}}{D_{\min}} &= n - 1 + \left(\frac{3}{4}\right)\frac{1}{1-4^{-n}} \\ \lim_{\epsilon\to 0} \frac{D_{\text{ave}}}{D_{\min}} &= \frac{n}{2(1-2^{-n})} \\ \lim_{\epsilon\to 0} \frac{D_{\max}}{D_{\text{ave}}} &= 2(1-n^{-1})(1-2^{-n}) + \left(\frac{3}{2}\right)\frac{1}{n(1+2^{-n})}. \end{aligned}$$

On the other hand, for a fixed bit error probability ϵ, letting $n \to \infty$ yields

$$\begin{aligned} \lim_{n\to\infty} \frac{D_{\max}}{D_{\min}} &= \frac{1}{2\epsilon} \\ \lim_{n\to\infty} \frac{D_{\text{ave}}}{D_{\min}} &= \frac{1}{2\epsilon} \\ \lim_{n\to\infty} \frac{D_{\max}}{D_{\text{ave}}} &= 1. \end{aligned}$$

Thus, for asymptotically large block lengths, the performance gain of a best index assignment over a worst index assignment or an average index assignment is $1/2\epsilon$, which can be very large. In this sense, a large fraction of index assignments can be considered "bad."

Corollary 1: For any fixed large n, as $\epsilon \to 0$ the relative MSE's of worst, average, and best index assignments for the uniform source obey the following ratios:

$$D_{\max} : D_{\text{ave}} : D_{\min} \approx 1 : 1/2 : 1/n$$

and for any fixed ϵ, as $n \to \infty$ the relative MSE's obey the ratios:

$$D_{\max} : D_{\text{ave}} : D_{\min} = 1 : 1 : 2\epsilon.$$

That is, for any $\epsilon > 0$, the expected distortion of a randomly chosen index assignment asymptotically equals (as the blocklength grows) that of the worst index assignment. If an integer chosen uniformly at random from $\mathcal{S}$ is normalized to have zero mean and unit variance, then the resulting distortions corresponding to the best, worst, and random index assignments are given by

$$\begin{aligned} \tilde{D}_{\min} &\triangleq \frac{D_{\min}}{\sigma_{\mathcal{S}}^2} = 4\epsilon \\ \tilde{D}_{\max} &\triangleq \frac{D_{\max}}{\sigma_{\mathcal{S}}^2} = 2(1-(1-2\epsilon)^{n-1}) + 3\epsilon\frac{(1-2\epsilon)^{n-1}}{1-4^{-n}} \\ \tilde{D}_{\text{ave}} &\triangleq \frac{D_{\text{ave}}}{\sigma_{\mathcal{S}}^2} = 2\frac{1-(1-\epsilon)^n}{1-2^{-n}}. \end{aligned}$$

Fig. 1 compares $\tilde{D}_{\min}$, $\tilde{D}_{\max}$, and $\tilde{D}_{\text{ave}}$. The horizontal line at normalized distortion 1.0 represents the distortion achievable with no information transmission (by simply reproducing the mean of the source at the receiver). Thus, the usefulness of any index assignment

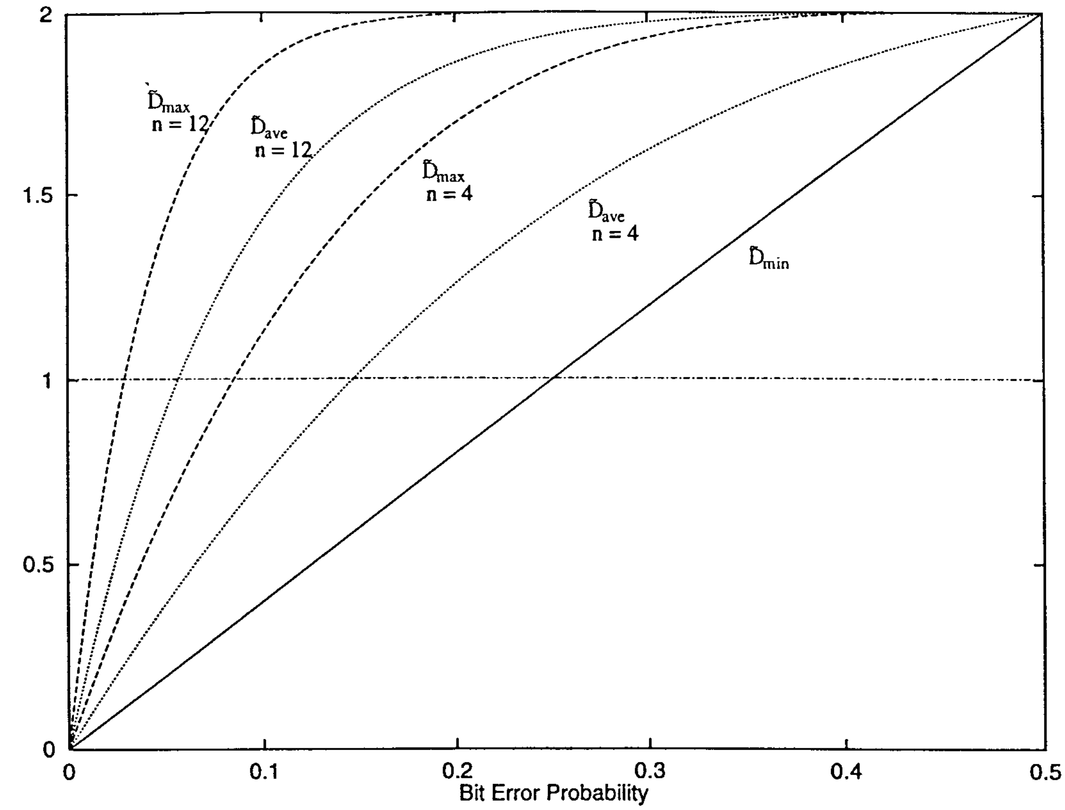

Fig. 1. The best, worst, and average performance achievable by index assignments for a uniform source. The solid line corresponds to $\tilde{D}_{\min}$. The dashed and dotted curves show $\tilde{D}_{\max}$ and $\tilde{D}_{\text{ave}}$, respectively, for $n = 4, 12$. The horizontal line represents the variance of the source, an achievable distortion at zero transmission rate.

is limited to values of ϵ smaller than the bit error probability determined by the intersection of this horizontal line and the distortion curve corresponding to the index assignment. Since $\lim_{n\to\infty} \tilde{D}_{\max} = \lim_{n\to\infty} \tilde{D}_{\text{ave}} = 2$ for any $\epsilon \in (0, 1/2]$, the useful region of bit error probabilities for the worst and average index assignments shrinks steadily as the blocklength increases. If $n\epsilon \ll 1$, then we obtain the approximations (linear in ϵ)

$$\tilde{D}_{\max} \approx \left(4(n-1) + \frac{3}{1-4^{-n}}\right)\epsilon \quad \text{and} \quad \tilde{D}_{\text{ave}} \approx \left(\frac{2n}{1-2^{-n}}\right)\epsilon.$$

These hold for small ϵ on the curves in Fig. 1 for which n is not too large. Suppose these linearized approximations hold and suppose that n is large, but not too large (i.e., if $2^{-n} \ll 1$ while maintaining $n\epsilon \ll 1$). Then, the useful regions of the worst and average index assignments can be approximated as $\epsilon \in (0, 1/(4n))$ and $\epsilon \in (0, 1/(2n))$, respectively. These intervals are obtained by examining which values of ϵ yield distortions less than 1. Note that $\tilde{D}_{\min} = 4\epsilon$ is independent of n and linear on the full range $\epsilon \in [0, 1/2]$. Thus, the useful region of the best index assignment is $(0, 1/4)$ irrespective of the blocklength n.

IV. Generalization to Vector Quantizers

The natural binary code was shown to minimize the distortion D for a uniform scalar quantizer and a uniform source in [10] and was generalized to a class of vector quantizers in [11]. The class of vector quantizers in [11] is the same class studied in [12] and [17]–[19] and was referred to in [12] as "binary lattice vector quantization." In contrast, we demonstrate by means of a counterexample that the distortion maximization property of the worst code for a uniform scalar quantizer cannot be generalized to arbitrary binary lattice vector quantizers. We do, however, show that the worst code maximizes the distortion among all *affine* index assignments for arbitrary binary lattice vector quantizers.

For any positive integer d, let R^d denote d-dimensional Euclidean space. We use a horizontal bar to distinguish between real vectors $\bar{\mathbf{x}} \in R^d$ and binary vectors $\mathbf{i} \in Z_2^n$. The Euclidean norm of a vector $\bar{\mathbf{x}} \in R^d$ is denoted by $\|\bar{\mathbf{x}}\|$.

Definition 6: A d-dimensional, 2^n-point *binary lattice vector quantizer* is a vector quantizer with code vectors of the form $\bar{\mathbf{y}}_i = \bar{\mathbf{y}}_0 + \sum_{l=0}^{n-1} \bar{\mathbf{v}}_l i_l$ for $i \in \mathcal{S}$, where $\bar{\mathbf{y}}_0 \in R^d$, and $\mathcal{V} = \{\bar{\mathbf{v}}_l\}_{l=0}^{n-1} \subset R^d$ is a *generating set* ordered by $\|\bar{\mathbf{v}}_0\| \le \|\bar{\mathbf{v}}_1\| \le \cdots \le \|\bar{\mathbf{v}}_{n-1}\|$.

Analogous to (2), the *channel distortion* of a binary lattice vector quantizer with equiprobable code vectors is defined as

$$D \triangleq 2^{-n} \sum_{i\in\mathcal{S}} \sum_{j\in\mathcal{S}} \|\bar{\mathbf{y}}_i - \bar{\mathbf{y}}_j\|^2 \rho_{\pi(\mathbf{i})+\pi(\mathbf{j})}. \tag{10}$$

A uniform scalar quantizer with step size Δ is a special case of a binary lattice vector quantizer with $d = 1$ and $\bar{\mathbf{v}}_l = 2^l\Delta$ for $l \in \{0, 1, \cdots, n-1\}$.

The results of Section III also apply for binary lattice quantizers if we replace $\eta(\mathbf{i})$ by $\bar{\mathbf{z}}(\mathbf{i}) = \bar{\mathbf{y}}_{\pi^{-1}(i)}$. In particular, Lemma 2 becomes

$$D = 2 \sum_{\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}} \|2^{-n}\hat{\bar{\mathbf{z}}}(\mathbf{a})\|^2 \left(1 - (1-2\epsilon)^{w(\mathbf{a})}\right) \tag{11}$$

and thus (3) becomes

$$4\epsilon \sum_{\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}} \|2^{-n}\hat{\bar{\mathbf{z}}}(\mathbf{a})\|^2 \le D \le 2(1-(1-2\epsilon)^n) \sum_{\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}} \|2^{-n}\hat{\bar{\mathbf{z}}}(\mathbf{a})\|^2. \tag{12}$$

We also have by (4) that

$$\sum_{\mathbf{a}\in Z_2^n\setminus\{\mathbf{0}\}} \|2^{-n}\hat{\bar{\mathbf{z}}}(\mathbf{a})\|^2 = 2^{-n}\sum_{i\in\mathcal{S}}\|\bar{\mathbf{y}}_i\|^2 - \left\|2^{-n}\sum_{i\in\mathcal{S}}\bar{\mathbf{y}}_i\right\|^2 = \frac{1}{4}\sum_{l=0}^{n-1}\|\bar{\mathbf{v}}_l\|^2 \tag{13}$$

for any choice of index assignment π. It is difficult, however, to find $\max_\pi \|2^{-n}\hat{\bar{\mathbf{z}}}(\mathbf{1})\|^2$ for an arbitrary index assignment. For affine index assignments, we have by (8)

$$\hat{\bar{\mathbf{z}}}(\mathbf{a}) = -2^{n-1}\sum_{l=0}^{n-1}\bar{\mathbf{v}}_l I_{\{\mathbf{a}\mathsf{G}^T=\mathbf{e}^{(l)}\}}$$

and thus

$$\max_{\pi \text{ affine}} \|2^{-n}\hat{\bar{\mathbf{z}}}(\mathbf{1})\|^2 = \frac{1}{4}\|\bar{\mathbf{v}}_{n-1}\|^2. \tag{14}$$

Using (13) and (14), the same argument that led to the upper bound in Theorem 1 yields the following corollary.

Corollary 2: The channel distortion of a binary lattice vector quantizer, with generators $\bar{\mathbf{v}}_0,\cdots,\bar{\mathbf{v}}_{n-1}$, followed by an affine index assignment and a binary symmetric channel with bit error probability $\epsilon \in [0, 1/2]$ satisfies

$$D \le \frac{1}{2}(1-(1-2\epsilon)^n)\|\bar{\mathbf{v}}_{n-1}\|^2 + \frac{1}{2}(1-(1-2\epsilon)^{n-1})\sum_{l=0}^{n-2}\|\bar{\mathbf{v}}_l\|^2$$

and the worst code achieves the upper bound with equality.

Note that from (14), Corollary 2 can be generalized to all (i.e., affine and nonaffine) index assignments if $\max_\pi \|2^{-n}\hat{\bar{\mathbf{z}}}(\mathbf{1})\|^2 = \frac{1}{4}\|\bar{\mathbf{v}}_{n-1}\|^2$. However, in general Corollary 2 cannot be generalized in this manner, as demonstrated in the following corollary.

Corollary 3: The worst code does not maximize the MSE of an arbitrary binary lattice vector quantizer over all index assignments for a binary symmetric channel.

Proof: We show by means of a counterexample that for some binary lattice vector quantizer there exists an index assignment π_X yielding a higher MSE than that of the worst code π_W. Specifically, define the nonaffine 3-bit index assignment π_X by

$$\pi_X(\mathbf{i}) = \begin{cases}\pi_W(100), & \text{if } \mathbf{i}=011\\ \pi_W(011), & \text{if } \mathbf{i}=100\\ \pi_W(\mathbf{i}), & \text{otherwise.}\end{cases}$$

The table given at the bottom of the page explicitly lists the index assignments π_W and π_X, along with the Hadamard transform vectors $\hat{\bar{\mathbf{z}}}_W$ and $\hat{\bar{\mathbf{z}}}_X$. The identity

$$\|\bar{\mathbf{u}}+\bar{\mathbf{w}}\|^2 + \|\bar{\mathbf{u}}-\bar{\mathbf{w}}\|^2 = 2\|\bar{\mathbf{u}}^2\| + 2\|\bar{\mathbf{w}}\|^2 \quad \bar{\mathbf{u}},\bar{\mathbf{w}}\in R^d$$

will be used frequently in what follows. Also, to simplify notation we set $\gamma = 1-2\epsilon$. By (11) the channel distortion of the worst code is

$$\begin{aligned}D_W &= 2(1-\gamma^3)\|(-4/8)\bar{\mathbf{v}}_2\|^2 + 2(1-\gamma^2)\\ &\quad\times(\|(-4/8)\bar{\mathbf{v}}_0\|^2 + \|(-4/8)\bar{\mathbf{v}}_1\|^2)\\ &= \frac{1}{2}[(\|\bar{\mathbf{v}}_2\|^2 + \|\bar{\mathbf{v}}_1\|^2 + \|\bar{\mathbf{v}}_0^2\|)\\ &\quad - \gamma^2(\|\bar{\mathbf{v}}_1\|^2 + \|\bar{\mathbf{v}}_0\|^2) - \gamma^3\|\bar{\mathbf{v}}_2\|^2]\end{aligned}$$

and the channel distortion of the index assignment π_X is

$$\begin{aligned}D_X &= 2(1-\gamma^3)\|(-2/8)(\bar{\mathbf{v}}_2+\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0)\|^2\\ &\quad + 2(1-\gamma^2)(\|(-2/8)(\bar{\mathbf{v}}_2-\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0)\|^2\\ &\quad + \|(-2/8)(\bar{\mathbf{v}}_2+\bar{\mathbf{v}}_1-\bar{\mathbf{v}}_0)\|^2)\\ &\quad + 2(1-\gamma)\|(2/8)(\bar{\mathbf{v}}_2-\bar{\mathbf{v}}_1-\bar{\mathbf{v}}_0)\|^2\\ &= \frac{1}{8}[4(\|\bar{\mathbf{v}}_2\|^2 + \|\bar{\mathbf{v}}_1\|^2 + \|\bar{\mathbf{v}}_0\|^2) - \gamma\|\bar{\mathbf{v}}_2 - (\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0)\|^2\\ &\quad - 2\gamma^2(\|\bar{\mathbf{v}}_2\|^2 + \|\bar{\mathbf{v}}_1-\bar{\mathbf{v}}_0\|^2) - \gamma^3\|\bar{\mathbf{v}}_2 + (\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0)\|^2].\end{aligned}$$

Thus, $D_X > D_W$ whenever

$$\begin{aligned}0 &> \gamma^2(\|\bar{\mathbf{v}}_2 + (\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0)\|^2 - 4\|\bar{\mathbf{v}}_2\|^2)\\ &\quad + 2\gamma(\|\bar{\mathbf{v}}_2\|^2 + \|\bar{\mathbf{v}}_1-\bar{\mathbf{v}}_0\|^2 - 2(\|\bar{\mathbf{v}}_1\|^2 + \|\bar{\mathbf{v}}_0\|^2))\\ &\quad + \|\bar{\mathbf{v}}_2 - (\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0)\|^2\\ &= \gamma^2[2(\|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0\|^2 - \|\bar{\mathbf{v}}_2\|^2) - \|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0-\bar{\mathbf{v}}_2\|^2]\\ &\quad - 2\gamma(\|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0\|^2 - \|\bar{\mathbf{v}}_2\|^2) + \|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0-\bar{\mathbf{v}}_2\|^2\\ &= ([2(\|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0\|^2 - \|\bar{\mathbf{v}}_2\|^2) - \|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0-\bar{\mathbf{v}}_2\|^2]\gamma\\ &\quad - \|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0-\bar{\mathbf{v}}_2\|^2)(\gamma-1).\end{aligned}$$

Hence, for any eight-point binary lattice vector quantizer satisfying

$$\|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0\|^2 > \|\bar{\mathbf{v}}_2\|^2 + \|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0-\bar{\mathbf{v}}_2\|^2 \tag{15}$$

the index assignment π_X is worse than the worst code if

$$\frac{\|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0-\bar{\mathbf{v}}_2\|^2}{2(\|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0\|^2 - \|\bar{\mathbf{v}}_2\|^2) - \|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0-\bar{\mathbf{v}}_2\|^2} < \gamma < 1$$

or equivalently, whenever

$$0 < \epsilon < \frac{(\|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0\|^2 - \|\bar{\mathbf{v}}_2\|^2) - \|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0-\bar{\mathbf{v}}_2\|^2}{2(\|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0\|^2 - \|\bar{\mathbf{v}}_2\|^2) - \|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0-\bar{\mathbf{v}}_2\|^2}. \tag{16}$$

In particular, if $\bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_0 = \alpha\bar{\mathbf{v}}_2$ for $\alpha > 1$, then (15) is satisfied and (16) reduces to

$$0 < \epsilon < \frac{2\|\bar{\mathbf{v}}_2\|}{\|\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0\| + 3\|\bar{\mathbf{v}}_2\|} = \frac{2}{\alpha+3}. \tag{17}$$

The right-hand side of (17) can be arbitrarily close to $1/2$ as $\alpha\to 1$. Thus, for any $\epsilon \in (0, 1/2)$, a binary lattice vector quantizer can be found for which the index assignment π_X is worse than the worst code. □

$\mathbf{i}$	$\pi_W(\mathbf{i})$	$\hat{\bar{\mathbf{z}}}_W(\mathbf{i})$	$\pi_X(\mathbf{i})$	$\hat{\bar{\mathbf{z}}}_X(\mathbf{i})$
000	000	$8\bar{\mathbf{y}}_0 + 4(\bar{\mathbf{v}}_0+\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_2)$	000	$8\bar{\mathbf{y}}_0 + 4(\bar{\mathbf{v}}_0+\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_2)$
001	101	$\bar{\mathbf{0}}$	101	$\bar{\mathbf{0}}$
010	110	$\bar{\mathbf{0}}$	110	$\bar{\mathbf{0}}$
011	011	$\bar{\mathbf{0}}$	111	$\bar{\mathbf{0}}$
100	111	$\bar{\mathbf{0}}$	011	$+2(\bar{\mathbf{v}}_2-\bar{\mathbf{v}}_1-\bar{\mathbf{v}}_0)$
101	010	$-4\bar{\mathbf{v}}_1$	010	$-2(\bar{\mathbf{v}}_2+\bar{\mathbf{v}}_1-\bar{\mathbf{v}}_0)$
110	001	$-4\bar{\mathbf{v}}_0$	001	$-2(\bar{\mathbf{v}}_2-\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0)$
111	100	$-4\bar{\mathbf{v}}_2$	100	$-2(\bar{\mathbf{v}}_2+\bar{\mathbf{v}}_1+\bar{\mathbf{v}}_0)$.

REFERENCES

[1] J. D. Marca and N. Jayant, "An algorithm for assigning binary indices to the code vectors of a multi-dimensional quantizer," in *Proc. IEEE Int. Conf. Communications*, Seattle, WA, June 1987, vol. 1, pp. 1128–1132.
[2] N. Farvardin, "A study of vector quantization for noisy channels," *IEEE Trans. Inform. Theory*, vol. 36, pp. 799–809, July 1990.
[3] H.-S. Wu and J. Barba, "Index allocation in vector quantisation for noisy channels," *Electron. Lett.*, vol. 29, pp. 1318–1320, July 1993.
[4] K. Zeger and A. Gersho, "Pseudo-gray coding," *IEEE Trans. Commun.*, vol. 38, pp. 2147–2158, Dec. 1990.
[5] P. Knagenhjelm and E. Agrell, "The Hadamard transform—A tool for index assignment," *IEEE Trans. Inform. Theory*, vol. 42, pp. 1139–1151, July 1996.
[6] D. J. Goodman and T. J. Moulsley, "Using simulated annealing to design transmission codes for analogue sources," *Electron. Lett.*, vol. 24, pp. 617–618, May 1988.
[7] Y. Yamaguchi and T. S. Huang, "Optimum binary fixed-length block codes," Quarterly Progress Rep. 78, MIT, Cambridge, MA, July 1965.
[8] T. S. Huang, "Optimum binary code," Quarterly Progress Rep. 82, MIT, Cambridge, MA, July 1966.
[9] T. S. Huang *et al.*, "Design considerations in PCM transmission of low-resolution monochrome still pictures," *Proc. IEEE*, vol. 55, pp. 331–335, Mar. 1967.
[10] T. R. Crimmins, H. M. Horwitz, C. J. Palermo, and R. V. Palermo, "Minimization of mean-square error for data transmitted via group codes," *IEEE Trans. Inform. Theory*, vol. IT-15, pp. 72–78, Jan. 1969.
[11] S. W. McLaughlin, D. L. Neuhoff, and J. J. Ashley, "Optimal binary index assignments for a class of equiprobable scalar and vector quantizers," *IEEE Trans. Inform. Theory*, vol. 41, pp. 2031–2037, Nov. 1995.
[12] A. Méhes and K. Zeger, "Binary lattice vector quantization with linear block codes and affine index assignments," *IEEE Trans. Inform. Theory*, vol. 44, pp. 79–95, Jan. 1998.
[13] R. E. Totty and G. C. Clark, "Reconstruction error in waveform transmission," *IEEE Trans. Inform. Theory*, vol. IT-13, pp. 336–338, Apr. 1967.
[14] A. Méhes and K. Zeger, "Redundancy free codes for binary discrete memoryless channels," in *Proc. Conf. Information Science and Systems*, Princeton, NJ, 1994, pp. 1057–1062.
[15] ——, "On the performance of affine index assignments for redundancy free source-channel coding," in *Proc. Data Compression Conf.*, Snowbird, UT, Mar. 1995, p. 433.
[16] ——, "Affine index assignments for binary lattice quantization with channel noise," in *ISIT-95*, Whistler, B.C., Canada, Sept. 1995, p. 377.
[17] R. Hagen and P. Hedelin, "Robust vector quantization by a linear mapping of a block code," *IEEE Trans. Inform. Theory*, vol. 45, pp. 200–218, Jan. 1999.
[18] ——, "Design methods for VQ by linear mappings of block codes," in *Proc. IEEE Int. Symp. Inform. Theory*, Trondheim, Norway, June 1994, p. 241.
[19] ——, "Robust vector quantization by linear mappings of block-codes," in *Proc. IEEE Int. Symp. Inform. Theory*, San Antonio, TX, Jan. 1993, p. 171.

Efficient Code Constructions for Certain Two-Dimensional Constraints

Roman Talyansky, Tuvi Etzion, *Member, IEEE*,
and Ron M. Roth, *Senior Member, IEEE*

***Abstract*—Efficient encoding algorithms are presented for two types of constraints on two-dimensional binary arrays. The first constraint considered is that of t-conservative arrays, where each row and each column has at least t transitions of the form '0' →'1' or '1' → '0.' The second constraint is that of two-dimensional DC-free arrays, where in each row and each column the number of '0's equals the number of '1's.**

***Index Terms*—Balanced arrays, conservative arrays, two-dimensional constraints, two-dimensional DC-free arrays, two-dimensional runlength-limited constraints.**

I. INTRODUCTION

Recent developments in optical storage—especially in the area of holographic memory—are attempting to increase the recording density by exploiting the fact that the recording device is a *surface*. Under this new model, the recorded data is regarded as two-dimensional (2-D), as opposed to the track-oriented one-dimensional (1-D) recording paradigm [5], [22]. The new approach, however, introduces new types of constraints on the data—those now become 2-D rather than 1-D.

One-dimensional constraints were extensively studied, and there are several known methodologies for designing codes for such constraints; see, for instance, [15], [16], and [24]. On the other hand, our knowledge of 2-D constraints is much less profound. This might be attributed in part to the fact that the practical interest in those constraints has been risen only recently; however, it seems that the main reason for such a lack of knowledge is the provable difficulty of 2-D constraints compared to the 1-D case [4], [23]. Nevertheless, there have been several results reported on 2-D runlength-limited coding [7], [8], coding for holographic memory [2], [3], [27], and multitrack modulation coding [14], [17], [18]. Reference [9] deals with the computation of the capacity of 2-D constraints, and bounds on the capacity of certain specific constraints are presented in [6] and [28].

We next describe briefly two applications of 2-D constrained codes: holographic storage and barcodes. In holographic recording, data is stored optically in the form of 2-D pages. Each data page is a pattern of '0's and '1's, represented by dark and light spots, respectively. What is actually stored is the interference pattern between an optical representation of the data page and a so-called reference beam. Several holograms can be stored in the same physical volume, each being encoded with a distinct reference beam. To increase the reliability of the holographic recording system, the patterns of '0's and '1's need to satisfy certain modulation constraints. One example of such a constraint is avoiding long periodic stretches of dark or light spots in both dimensions [5], [27]. In addition, it is desirable to use coding techniques that do not permit a large imbalance

Manuscript received March 8, 1998. This work was supported in part by the United States—Israel Binational Science Foundation (BSF), Jerusalem, Israel, under Grant 95-522.

R. Talyansky is with the IBM Haifa Research Laboratory, Matam—Advanced Technology Center, Haifa 31905, Israel.

T. Etzion is with the Computer Science Department, Technion—Israel Institute of Technology, Haifa 32000, Israel.

R. M. Roth is with the Computer Science Department, Technion—Israel Institute of Technology, Haifa 32000, Israel (e-mail: ronny@cs.technion.ac.il).

Communicated by A. Barg, Associate Editor for Coding Theory.

Publisher Item Identifier S 0018-9448(99)01420-0.