# RECENT TRENDS IN LOSSY SOURCE CODING*

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**A survey is given on some recent areas of interest in lossy source coding, that is, source coding relative to a fidelity criterion. In particular we discuss some recent results in rate-distortion theory, universal coding, and vector quantization techniques, and we indicate some current open problems.**

## 1. INTRODUCTION

Lossy source coding (also called source coding with a fidelity criterion) is customarily considered a branch of information theory. The theory of lossy source coding was initiated by Claude Shannon, the founder of the principles of information theory [41]. In this paper we shall survey some aspects of lossy source coding which have developed into a separate field of study since Shannon's pioneering paper [42].

The survey will concentrate on theoretical issues rather than design and implementation problems. Our intention is to focus on some interesting recent results and open problems instead of giving an exhaustive survey of the existing theory and research trends. The topics we discuss here strongly reflect our own research interests. More comprehensive material on the practical side of lossy source coding can be found in e.g. [2], [13], and Kieffer [18].

It is assumed that the reader is familiar with the basic definitions and results of information theory, such as given in Cover and Thomas [9], Csiszár and Fritz [10] or Linder and Lugosi [24] (the last two books are in Hungarian).

Lossy source coding is concerned with the theoretical problems of signal compression: data is given by a mathematical model and is to be encoded into strings of binary digits so that both the encoding rate (the number of bits per data sample) and the distortion (a given measure of dissimilarity between the original signal and the signal reconstructed from its coded form) satisfy some prescribed constraints. The coding rate and the reconstruction quality are two conflicting measures of performance, and the main issue is to find an efficient tradeoff between these quantities. The abstract "data" can model different real-life signals such as speech, audio, still images, or video, and an efficient digital representation can serve several purposes such as transmission over a digital channel, storage on digital media, or easing the computational burden for data encryption. The methods to achieve the desired compression ratio with only a permissible degradation in the reproduction quality are rather different from the methods of lossless (also called noiseless) source coding, where one wants to reduce the data rate without introducing any distortion in the reproduction, as in the compression of data files for storage on computers. Nevertheless, there is much interaction between these two fields, and a familiarity with the basics of lossless coding is an important prerequisite for studying lossy coding.

The paper is organized as follows. In Section 2 we introduce the basic model of an information source, and the concept of source coding, and then state some fundamental results of rate-distortion theory. Section 3 addresses the problem of universal source coding, i.e., source coding with no prior knowledge of the statistical properties of the source to be encoded. In Section 4 we examine the theoretical aspects of multidimensional signal quantizers, and in Section 5 we shall give a brief review of high-resolution quantization theory, the only method available to date to obtain analytical expressions for vector quantizer performance.

## 2. RATE-DISTORTION THEORY

The term "rate-distortion theory" refers to the branch of information theory dealing with the rate-distortion tradeoff in source coding when the encoded blocklength gets large, i.e., when we effectively assume that the message to be encoded has infinite length. The mathematical model of an information source we use consists of a sequence $X_1, X_2, \ldots, X_n, \ldots$ of random variables taking values in a set $\mathcal{S}$ called the source alphabet. We assume a source is completely described by its finite dimensional probability distributions. In this paper we usually take $\mathcal{S} = \mathbb{R}$, the real line, or sometimes $\mathcal{S} = \mathbb{R}^k$, i.e. $k$-dimensional Euclidean space. For example, the $X_i$ could be obtained by sampling from a "continuous time" signal $X(t)$: $X_i = X(iT)$ for some $T > 0$. Often, however, the signal to be encoded needs no time-discretization, as with compression of digital images.

A *source code of blocklength* $n$ consists of an *encoder* $g$ and a *decoder* $\varphi$. The encoder maps $n$-dimensional vectors into a finite set of all binary strings:

$$g : \mathbb{R}^n \to \{s_1, \ldots, s_N\} \subset \{0,1\}^*,$$

where $\{0,1\}^*$ denotes the set of finite length binary sequences. The decoder is a mapping

$$\varphi : \{s_1, \ldots, s_N\} \to \{y_1, \ldots, y_N\} \subset \mathbb{R}^n,$$

where the $y_i$ are called the reproduction (or code) vectors. When the binary sequences $s_1, \ldots, s_N$ have the same length, we say that $(g, \varphi)$ is a *fixed rate* code. We assume that given the set of codevectors $\{y_1, \ldots, y_N\}$ the encoder of a fixed rate code uses binary strings of the minimum possible length such that the mapping $\varphi(g(\cdot)) : \mathbb{R}^n \to \mathbb{R}^n$ is unchanged. The pair $(g, \varphi)$ is called a *variable length* code when $\{s_1, \ldots, s_N\}$ is a binary prefix

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code i.e., when no $s_i$ is a prefix of any $s_j$ for $i \neq j$. In both cases, for any $k \geq 1$ there is one and only one way the binary sequence

$$g(X_1, \ldots, X_n) \ldots g(X_{(k-1)n+1}, \ldots, X_{kn}),$$

obtained by concatenating the binary strings $g$ assigns to the vectors $(X_{(i-1)n+1}, \ldots, X_{in})$, $i = 1, \ldots, k$, can be decomposed into binary strings from the set $\{s_1, \ldots, s_N\}$. That is, the code is uniquely decodable. Thus, if the code $(g, \varphi)$ is used $k$-times to encode the first $kn$ samples of the source, and the resulting binary sequence is transmitted over an error free channel, then the reproduction vectors $y_i = \varphi(g(X_{(i-1)n+1}, \ldots, X_{in}))$, $i = 1, \ldots, k$ can be recovered without error. However, distortion is introduced by representing an $n$-dimensional vector (which possibly can take infinitely many values) by a finite set of vectors.

The *sample distortion* between two vectors $x^n = (x_1, \ldots, x_n)$ and $y^n = (y_1, \ldots, y_n)$ is measured by the squared error per sample

$$d_n(x^n, y^n) = \frac{1}{n} \sum_{i=1}^{n} (x_i - y_i)^2.$$

We use squared error here for the sake of simplicity — all the results in this section are valid with more general measures of distortion. The distortion of the code is given as the expected value of the sample distortion whenever $X^n = (X_1, \ldots, X_n)$ is encoded:

$$\Delta(g, \varphi) = \mathbf{E} d_n[X^n, \varphi(g(X^n))].$$

The above distortion is of course finite when $\mathbf{E}\|X^n\|^2 < \infty$, where $\|\cdot\|$ denotes Euclidean norm. The dependence of the distortion on the distribution of the source is suppressed in the notation. The rate $R(g)$ of a fixed rate code is defined as

$$R(g, \varphi) = \frac{1}{n} \log N,$$

i.e., the base 2 logarithm of the number of reproduction vectors normalized by the blocklength. Note that this is the number of encoding bits per source sample. The rate of a variable length code is the expected value of the normalized length of the encoded binary string:

$$R(g, \varphi) = \frac{1}{n} \mathbf{E}\left(\text{length}[g(X^n)]\right).$$

It is well known (see e.g. [10]) that the encoder always can be chosen (without changing the mapping $\varphi(g(\cdot))$) so that the average codelength of a variable length code is within $1/n$ bits of its lower bound $H(\varphi(g(X^n)))$, the entropy of the reproduction.

It is intuitively clear that both the rate and the distortion can not be arbitrarily small at the same time; if one of them is small then the other quantity will inevitably increase. In the remainder of this section we consider only fixed rate codes and present fundamental results on the distortion rate tradeoff. The main problem is the characterization of the minimum distortion that fixed rate codes can achieve while having rate that is less than or equal to a given rate $R$. For reasons of practicality, we assume the code rate is constant, but we note that analogous results hold when the distortion is fixed and the question is to find the minimum achievable rate.

Define $\hat{D}_n(R)$ as the minimum distortion which can be achieved using an $n$-length fixed rate source code of rate not exceeding $R > 0$, i.e.,

$$\hat{D}_n(R) = \inf_{R(g,\varphi) \leq R} \Delta(g, \varphi). \qquad (1)$$

When $X_1, X_n, \ldots$ is a stationary sequence of random variables, it is not hard to see that

$$\lim_{n \to \infty} \hat{D}_n(R) = \inf_{n \geq 1} \hat{D}_n(R) \stackrel{\text{def}}{=} \hat{D}(R). \qquad (2)$$

The function $\hat{D}(R)$ is a lower bound on the distortion of any fixed rate source code of rate at most $R$. The quantity $\hat{D}(R)$ is often called the *operational distortion-rate* function of the source with respect to fixed rate coding. Also, it follows from the first equality in (2) that for a given $\epsilon > 0$, if $n$ is large enough there always exists a code $(g_n, \varphi_n)$ with $R(g_n, \varphi_n) \leq R$ and $\Delta(g_n, \varphi_n) < \hat{D}(R) + \epsilon$. Define the $n^{th}$ *order distortion-rate function* by

$$D_n(R) = \inf\{\mathbf{E} d_n(\bar{X}^n, \bar{Y}^n) : n^{-1} I(\bar{X}^n, \bar{Y}^n) \leq R\}$$

where the infimum is taken over all pairs of $(\bar{X}^n, \bar{Y}_n)$ such that $\bar{X}^n$ and $X^n$ have the same distribution and the mutual information [10] between $\bar{X}^n$ and $\bar{Y}^n$ is at most $R$. By elementary properties of the mutual information we have

$$\hat{D}_n(R) \geq D_n(R)$$

for all $n$. The *source coding theorem* for ergodic sources states (see Berger [4]) that if $X_1, X_2, \ldots$ is stationary and ergodic, then

$$\hat{D}(R) = \lim_{n \to \infty} D_n(R) \stackrel{\text{def}}{=} D(R).$$

The quantity $D(R)$ is the *distortion-rate function of the source*. This theorem was first stated by Shannon [42] for finite alphabet memoryless sources, i.e., when $X_1, X_2, \ldots$ is sequence of independent and identically distributed random variables which can take only finitely many values. The reason why this result is of fundamental importance is that it gives a characterization of the otherwise intractable quantity $\hat{D}(R)$. On the other hand, several properties of $D(R)$ are known as a function of $R$ (see e.g. Berger [4]).

If we relax the ergodic assumption and require only stationarity, we find that $\hat{D}(R) > D(R)$ can happen (see Gray and Davisson [15]). On the other hand, using variable length codes in the definition of $\hat{D}(R)$ instead of fixed rate codes, we have $\hat{D}(R) = D(R)$ for stationary nonergodic sources by a result of Leon-Garcia *et al.* [21]. We note here that several variations of the source coding theorem exist, proving similar results for more general source and reproduction alphabets, distortion measures, and source distributions (see [18] for more references).

The above discussion illustrates the advantage of using source codes of large blocklength. In general, however, block codes of length $n$ introduce a delay of order $n$, since unless the code has special structure, the encoder has to wait until the last sample $X_n$ arrives before producing the binary codeword for $X^n$. Also, the complexity of encoding and decoding critically depend on the blocklength. Therefore, it is of great interest to estimate what improvement in distortion we can expect by increasing $n$. Unfortunately, an exact calculation of $\hat{D}_n(R)$, the distortion of the best code of length $n$ and rate $R$ seems intractable. Thus we have to settle for asymptotic results. Pilc [36] showed that for a finite alphabet memoryless source

$$\hat{D}_n(R) - D(R) \le c\left(\frac{\log n}{n}\right),$$

where $c$ is a constant depending on the source distribution and the rate. Wyner [45] proved the same asymptotics for memoryless Gaussian sources, and later obtained [46] that

$$\hat{D}_n(R) - D(R) \le c\sqrt{\frac{\log n}{n}}, \tag{3}$$

for any stationary Gaussian source whose spectral density satisfies some smoothness conditions. Linder *et al.* [26] showed that (3) holds when the source is memoryless and the $X_i$ are bounded but otherwise have arbitrary distribution (i.e. this includes infinite alphabet sources). It is presently unknown whether Pilc's result is sharp, that is whether there exists a lower bound of the type $n^{-1}\log n$, or whether his upper bound can be generalized to memoryless sources with continuous distribution. We note that far more and stronger results are known for lossless coding (see e.g. Krichevsky and Trofimov [19]).

## 3. UNIVERSAL CODING

Results of rate-distortion theory guarantee the existence of source codes performing near the achievable optimum. These codes however strongly rely on the knowledge of the probability distribution of the source. In practice, the distribution is typically unkown, therefore there is a strong demand for coding methods that "learn" good codes from observing data emitted by the source.

The term "universal code" is used to denote a sequence of lossy codes of increasing blocklength such that they perform near-optimally on each source from a given collection of sources. Let $\mathcal{X}$ be a family of stationary real sources. For any source $\boldsymbol{X} = X_1, X_2, \ldots$ in $\mathcal{X}$ we define the operational distortion-rate function $\hat{D}(R, \boldsymbol{X})$ and the distortion-rate function $D(R, \boldsymbol{X})$ of $\boldsymbol{X}$ as in the previous section, only here we make explicit the dependence of these quantities on the particular source. A sequence of fixed rate codes $(g_n, \varphi_n)$, $n = 1, 2, \ldots$ is said to be *weakly universal* at rate $R$ if

$$R(g_n, \varphi_n) \le R$$

and

$$\lim_{n\to\infty} \mathbf{E}d_n[X^n, \varphi(g(X^n))] = \hat{D}(R, \boldsymbol{X})$$

for all $\boldsymbol{X} \in \mathcal{X}$. In other words, $(g_n, \varphi_n)$ performs optimally for all sources in $\mathcal{X}$ when $n \to \infty$. *Strong universality* means that in the above limit we have uniform convergence over $\mathcal{X}$. The practical significance of universal codes is clear: the same code can be used for different sources without much degradation in performance. Theoretically, the proof of existence and/or the construction of universal codes is often very challenging.

The existence of fixed rate weak universal codes for the class of stationary sources was first proved by Ziv [54] for general source and reproduction alphabets and distortion measures that include real sources and the mean-squared error criterion considered here. Neuhoff *et al.* [33] and Kieffer [16] provided various generalizations, the latter also dealing with universal variable-length lossy and lossless coding. Analyzing Ziv's scheme Linder *et al.* [26] proved that for any $R > 0$ there exists a sequence of fixed rate codes $(g_n, \varphi_n)$ of rate $R$ which are weakly universal for the family of real stationary sources with finite second moment, and for any memoryless source $\boldsymbol{X}$ with a bounded support

$$\mathbf{E}d_n[X^n, \varphi(g(X^n))] - \hat{D}(R, \boldsymbol{X}) \le c\sqrt{\frac{\log\log n}{\log n}},$$

where $c$ depends on the source and on $R$. In a related work [25] the same authors investigated fixed rate universal coding of memoryless sources. They showed that for the class of memoryless sources over a finite alphabet there exists a sequence of fixed rate codes $(g_n, \varphi_n)$ of rate $R$ such that

$$\mathbf{E}d_n[X^n, \varphi(g(X^n))] - \hat{D}(R, \boldsymbol{X}) \le c\left(\frac{\log n}{n}\right), \tag{4}$$

for some constant $c$. It was also shown that the code construction for proving the above result can be extended to yield a universal scheme for which

$$\mathbf{E}d_n[X^n, \varphi(g(X^n))] - \hat{D}(R, \boldsymbol{X}) \le c\sqrt{\frac{\log n}{n}}$$

for any bounded *real valued* memoryless source. Furthermore, the above rate of convergence also holds with probability one:

$$d_n[X^n, \varphi(g(X^n))] - \hat{D}(R, \boldsymbol{X}) \le$$
$$\le O\left(\sqrt{\frac{\log n}{n}}\right), \text{ with probability one,}$$

where $f(n) = O(h(n))$ means that $|f(n)| \le c|h(n)|$ for some $c > 0$, if $n$ is large enough. Yu and Speed [48] obtained a result similar to (4); they demonstrated the existence of a sequence of variable length codes such that for all memoryless sources over a finite alphabet,

$$d_n[X^n, \varphi(g(X^n))] \le D$$

and

$$R(g_n, \boldsymbol{X}) - R(D, \boldsymbol{X}) \le c\frac{\log n}{n},$$

where $R(g_n, \boldsymbol{X})$ is the expected codelength (which depends on the source) and $R(D, \boldsymbol{X})$ is the rate-distortion function of $\boldsymbol{X}$, the inverse of $D(R, \boldsymbol{X})$.

It is of great theoretical as well as practical interest to find universal codes with reasonable complexity. The above results, even though proved by code construction, do not provide viable implementation. A currently "hot" research topic is to find a lossy counterpart of the lossless universal Lempel-Ziv codes [55]. This problem is still unsolved, although progress towards this goal was made for such a construction by Steinberg and Gutman [43]. A computationally efficient universal lossy coding algorithm which sequentially updates the set of codevectors was given by Zhang and Wei [53].

## 4. VECTOR QUANTIZATION

Source coding theorems guarantee the existence of lossy source codes whose performance approaches the distortion-rate bound as the blocklength $n$ increases. While these results provide a beautiful theory, and indicate the best performance one can expect, their practical usefulness is limited by two facts. On the one hand, in practice, the complexity of the encoder and the decoder are limited by the computational resources available, so codes even with moderately large blocklengths cannot be realized. On the other hand, source coding theorems provide no guidance as to what optimal encoders look like. The theory of vector quantization is concerned with the design of encoders — i.e., vector quantizers — of fixed blocklength.

The basic problem may be formulated as follows. A $k$ dimensional, $N$ level vector quantizer $Q$ is a function of the form $Q : \mathbb{R}^k \to \{y_1, \ldots, y_N\}$, where $y_1, \ldots, y_N \in \mathbb{R}^k$ are the *reproduction points* or *codevectors*. A quantizer is defined by its reproduction points, and *quantization regions* $B_i = \{z : Q(z) = y_i\}$, $i = 1, \ldots, N$. $Q(z)$ is interpreted as the quantized value of an input vector $z \in \mathbb{R}^k$. Note that a $k$-length fixed rate source code $(g, \varphi)$ as defined in Section 2 determines a vector quantizer by $Q(z) = \varphi(g(z))$, and conversely, by encoding the output of a vector quantizer using fixed length binary sequences, we obtain a fixed rate source code. If $Z \in \mathbb{R}^k$ is a vector-valued random variable, the *average distortion* of $Q$ (with respect to $Z$) is defined as

$$\Delta(Q) = \mathbf{E}\frac{1}{k}\|Z - Q(Z)\|^2,$$

which is finite provided that $\mathbf{E}\|Z\|^2 < \infty$. (We assume $\mathbf{E}\|Z\|^2 < \infty$ throughout.) We seek vector quantizers with small distortion. A quantizer $Q^*$ is called *optimal*, if $\Delta(Q^*) \leq \Delta(Q)$ for any quantizer $Q$. Existence of optimal quantizers is always guaranteed, though they do not have to be unique (e.g. see Pollard [37] and Abaya and Wise [1]).

Some thought should convince the reader that any optimal quantizer satisfies the following properties.

a) NEAREST NEIGHBOR PROPERTY:

if $\|z - y_i\| < \|z - y_j\|$ for all $j$, then $Q^*(z) = y_i$ (ties may be broken arbitrarily).

b) CENTROID PROPERTY:

$$y_i = \mathbf{E}(Z | Z \in B_i), \quad i = 1, \ldots, N.$$

Property (a) says that input points should always be quantized to the nearest reproduction point. By the second property, the reproduction points should be placed at the centroids of the sets of points that are quantized to the same value. Apart from these properties, very little is known about optimal quantizers. Even for special distributions of $Z$, such as Gaussian or uniform distribution on the unit cube, no explicit formulas are available for the form of optimal quantizers. Using the above properties, various versions of an iterative method for designing quantizers have been introduced. This iterative method is varyingly known as the Lloyd-Max algorithm, Lloyd algorithm [28], Max algorithm [30], generalized Lloyd algorithm, or the Linde-Buzo-Gray algorithm [22]. The basic algorithm starts with an arbitrary quantizer, adjusts its regions $B_i$ first to satisfy Property (a), then adjusts its reproduction points $y_i$ to satisfy Property (b). Then these two steps are repeated for the re-adjusted quantizer. It is easy to see that the distortion of the quantizers obtained in successive stages of this algorithm cannot increase, therefore, it converges. Unfortunately, in general it does not converge to the distortion of an optimal quantizer, in other words, the algorithm may get stuck in local optima. In some lucky cases, however, the Lloyd-Max algorithm converges to an optimal quantizer; see Kieffer [17] and Trushkin [44] for sufficient conditions on the distribution of $Z$ for global optimality.

A common serious problem for the designer of a vector quantizer is that the distribution of the source $Z$ is unknown. The only information available is a *training sequence* $Z_1, \ldots, Z_m$ of vectors, where the $Z_i$'s can often be modeled efficiently as independent, identically distributed random variables, with the same distribution as $Z$. Then the designer generally measures the *empirical distortion*

$$\Delta_m(Q) = \frac{1}{m}\sum_{i=1}^{m}\frac{1}{k}\|Z_i - Q(Z_i)\|^2$$

of a quantizer $Q$, and tries to find a quantizer minimizing $\Delta_m(Q)$. Denote such an *empirically optimal* quantizer by $Q^*_m$, that is,

$$Q^*_m = \arg\min_Q \Delta_m(Q).$$

The distortion of such an empirically chosen quantizer is

$$\Delta(Q^*_m) = \mathbf{E}\left(\frac{1}{k}\|Z - Q^*_m(Z)\|^2 | Z_1, \ldots, Z_m\right).$$

Note that $\Delta(Q^*_m)$ is a random variable, as it depends on the (random) training sequence. One expects that if the training sequence is long enough, then the distortion of an empirically optimal quantizer gets close to the distortion of a truly optimal quantizer. Indeed, Pollard [37], [39] proved that

$$\Delta(Q^*_m) - \Delta(Q^*) \to 0, \text{ with probability one}$$

is true for any distribution of $Z$ with $\mathbf{E}\|Z\|^2 < \infty$. Under more restrictive conditions on the distribution, Pollard [38] also proved a central limit theorem, which indicates how fast the above difference can tend to zero. Along this line, Linder *et al.* [26] proved a large-deviation type probability inequality, which bounds the finite-sample behavior of the above difference. For example, the inequality in [26] implies that

$$\Delta(Q^*_m) - \Delta(Q^*) = O\left(\sqrt{\frac{\log m}{m}}\right) \text{ with probability one,}$$

which is true for any $Z$ with a bounded support. Apparently, the exact rate of convergence is still an unknown and challenging problem.

While in principle, empirically optimal quantizers can be found, the computational complexity of such a general algorithm is often too large for practical realizations. Many successful techniques have been proposed in the literature for designing quantizers from empirical data. We refer the reader to Makhoul *et al.* [29] Gersho and Gray [12] for good summaries of such algorithms. One of the most widely used of these techniques is the empirical version of the Lloyd-Max algorithm (also known as the Linde-Buzo-Gray algorithm, see [22]) which is simply the Lloyd-Max algorithm executed on the empirical distribution. This method produces good, but not necessarily empirically optimal quantizers from training sets. Sabin and Gray [40] demonstrated that if the size of the training sequence increases, this algorithm performs eventually as well as if the true distribution of the training data were known.

There have been some attempts to tackle the problem of having iterative descent algorithms get trapped in locally optimal solutions by introducing random perturbations or performing "soft competitions" in the iteration steps. Examples of these techniques are given in Yair *et al.* [47] Zeger *et al.* [52].

It is apparent that the encoding complexity of an unstructured vector quantizer (such as the typical output of the Lloyd-Max algorithm) becomes very quickly prohibitive when the vector dimension increases. There has been much research activity on finding good vector quantizers with structural constraints that ensure efficient implementation. Although there are several competing schemes such as trellis and tree structured quantizers, and lattice quantizers (see Gersho and Gray [12] and the references therein), so far none of these methods have been rigorously shown to achieve the rate-distortion limits. Among these schemes the theoretically best understood are lattice quantizers. As the next section will indicate, lattice quantizers can perform near the rate distortion limit and are conjectured to be optimal in a certain asymptotical sense.

## 5. HIGH-RATE QUANTIZATION THEORY

In all the previous sections we mainly dealt with $\hat{D}_n(R)$, the distortion of the best $n$-length fixed rate block code at rate $R$. We saw that rate-distortion theory characterizes the limit of this quantity as $n$ gets large, and that the theory of vector quantizer design helps estimate the error made when the optimal source coder is designed from empirical data. However, the very important question of what the value of $\hat{D}_n(R)$ is for a given source, dimension, and distortion measure, remains unanswered. We now describe some results from the high rate theory of quantization which comes the closest to solving this problem. Also, in the framework of this theory it is often possible to deduce relevant properties of optimal quantizers and source coding schemes using some structural constraint.

### 5.1. Resolution-Constrained Quantization

Let us consider a $k$-dimensional vector quantizer with $N$-codevectors as in Section 4. The dimension $k$ will be fixed throughout this section. Given the random vector $X^k$, we denote the mean-squared distortion of the best such quantizer by $D_r(N)$, i.e.:

$$D_r(N) = \min_Q \|X^k - Q(X^k)\|^2,$$

where the minimum is taken over $Q$'s with $N$ codevectors. As we mentioned before, there are no known methods for explicitly computing $D_r(N)$. As is turns out, however, for $N$ large enough there exist good approximations to $D_r(N)$. For $k = 1$ Bennett [3] and Panter and Dite [35] derived a formula for sources with "nice" densities

$$D_r(N) \sim \frac{1}{12} c(f) N^{-2}, \tag{5}$$

where $c(f)$ is a constant depending only on the source density $f$, and $\sim$ means that the ratio of the two sides approaches 1 when $N \to \infty$. This asymptotic formula provides a good approximation to $D_r(N)$ for $N$ large, and it has been observed (see e.g. Neuhoff [32]) that in many important cases (5) is quite accurate for $N \geq 8$. The first rigorous proof of (5) as well as its generalization to vector quantizers was given by Zador [49], [50]. He proved for any dimension $k$ that for sources with sufficiently smooth densities

$$D_r(N) \sim \alpha_k c(f) N^{-2/k}, \tag{6}$$

where $\alpha_k$ is a constant depending only on the dimension $k$, and $c(f)$ is an easily computable function of the source density. Bucklew and Wise [5] proved that Zador's formula (6) holds for any source $X^k$ having a density and satisfying $\mathbf{E}\|X^k\|^{2+\epsilon} < \infty$ for some $\epsilon > 0$. Cambanis and Gerr [6] investigated Bennett's "companding quantization", a scheme in which a memoryless mapping and its inverse is applied to the source and to the quantizer output, respectively, to implement scalar quantizers. They determined a simple sequence of $N$-level scalar quantizers which are asymptotically optimal in the sense that their distortion satisfies (5). The exact performance of companding quantizers under general conditions was investigated by Linder [23]. Na and Neuhoff [31] gave a multidimensional Bennett-type formula for certain sequences of quantizers. Recently, high-rate techniques have been applied to the performance analysis of quantizers with structural constraints by Neuhoff and Lee ([34] and [20]). Another application of the high-rate theory is an analysis of a universal quantization scheme by Zeger *et al.* [51].

### 5.2. Entropy-Constrained Quantization

The entropy $H(Q)$ of a quantizer with $N$ codevectors is defined as

$$-\sum_{i=1}^{N} \mathbf{P}\{Q(X^k) = y_i\} \log \mathbf{P}\{Q(X^k) = y_i\}.$$

Let $D_e(H)$ be the minimum distortion over all $k$-dimensional quantizers with entropy less than or equal to $H$:

$$D_e(H) = \min_{Q: H(Q) \leq H} E\|X^k - Q(X^k)\|^2,$$

where the minimum is taken over all quantizers with a finite number of levels. As we mentioned before, for any $Q$ there exists a variable length code $(g, \varphi)$ such that $Q(x) = \varphi(g(x))$ for all $x \in \mathbb{R}^k$ and

$$R(g, \varphi) \leq \frac{1}{n} H(Q) + \frac{1}{n}. \tag{7}$$

Since $R(g, \varphi) \geq \frac{1}{n} H(Q)$ always holds when $Q(x) = \varphi(g(x))$, the inequality (7) shows that $D_e(H)$ is very closely related to the distortion of optimal variable length source codes. On the other hand, $D_e(H)$ yields to high-rate analysis. Gish and Pierce [14] recognized that uniform scalar quantizers (i.e. quantizers with codepoints equally spaced along the real line) have good entropy constrained performance. Let $Q_u$ be a uniform quantizer with entropy $H(Q_u)$ and distortion $\Delta(Q_u)$. Gish and Pierce proved that under some conditions on the source density, as $H(Q_u) \to \infty$

$$\Delta(Q_u) \sim \frac{1}{12} 2^{2h(f)} 2^{-2H(Q_u)}, \qquad (8)$$

where $h(f) = \int f(x) \log f(x) dx$ is the differential entropy of the source density. They also argued that uniform quantizers are asymptotically optimal in one dimension in the sense that $D_e(H(Q_u))/\Delta(Q_u) \to 1$ as $H(Q_u) \to \infty$. Zador [49], [50] showed that for $k \geq 1$ the optimal entropy constrained quantizers have the asymptotics

$$D_e(H) \sim \beta_k 2^{2h(f)/k} 2^{-2H/k},$$

where $\beta_k$ is a constant that depends only on $k$. Based on a heuristic argument Gersho [11] conjectured that $\alpha_k = \beta_k$, and also that their value is the minimum normalized moment of inertia that a polytope capable of tessellating $\mathbb{R}^k$ can have. This conjecture is widely believed to be true, but no proof of it is known to date. Linder and Zeger [27] made precise and proved under general conditions Gersho's formula for the high-rate asymptotics of tessellating quantizers.

The multidimensional generalizations of uniform quantizers are lattice quantizers [8]. The code points of a $k$-dimensional lattice quantizer are the integral linear combinations of a given basis of $\mathbb{R}^k$. Lattice quantizers are popular because efficient algorithms are often known for their implementation (see e.g. Conway and Sloane [7]). Also, lattice quantizers based on "good" lattices have entropy-constrained performance near the optimum [11] and it may be conjectured that the rate distortion limit can be arbitrarily approached by lattice quantizers.

## 6. OPEN PROBLEMS

We mention a few open problems in the areas covered in this survey. In rate distortion theory the simple characterization of achievable rate and distortion levels for networks including several sources and receivers is unsolved. Also, efficient bounds are sought for the rate-distortion functions of non-Gaussian sources with memory. Computationally efficient universal codes and sharp redundancy bounds are hot research topics. The design of globally optimal quantizers at least for some often used vector source models is an open problem which has been unsuccessfully attacked in the past. An undecided question is whether tree-structured quantizers can achieve (at least in an asymptotic sense) the rate-distortion bound. A theory of high rate entropy constrained quantization as rigorous as the resolution-constrained theory is missing. A very interesting question is whether there exist lattice quantizers not using variable length encoding which have near optimum performance.

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