A comment regarding: "On the Capacity of Two-Dimensional Run Length Constrained Channels" *

A. Kato † and Kenneth Zeger ‡

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On page 1529, column 1, the first sentence after equation (10) says:

> "It is interesting to note that the one-dimensional capacity $E_{0,k}$ is known to converge to one (as k grows) at the rate $(\frac{1}{4}\log_2 e)/2^k$."

Since no proof or citation for this fact was given in the paper, we elaborate here on why this is true.

Proof. It is known (as noted on page 1527, column 1) that the capacity can be written as

$$E_{0,k} = \log_2 \hat{X}_k$$

where $\hat{X}_k$ is the largest real root of the polynomial

$$X^{k+1} - X^k - X^{k-1} - \cdots - X - 1$$

or equivalently the largest real root of

$$f(X) = X^{k+1}(X-2) + 1.$$

The derivative of f is

$$f'(X) = X^k \left((k+2)X - 2(k+1)\right)$$

so

$$f'(X) > 0 \Longleftrightarrow X > 2 - \frac{2}{k+2}.$$

For each $\alpha \geq 0$ and each $k \geq 1$, let

$$X_{k,\alpha} = 2(1 - \alpha 2^{-k-2}).$$

We have

$$f(X_{k,\alpha}) = 1 - \alpha(1 - \alpha 2^{-k-2})^{k+1}.$$

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*A. Kato and K. Zeger, *IEEE Trans. on Information Theory*, vol. 45, no. 4, pp. 1527-1540, July 1999.

†A. Kato is deceased.

‡K. Zeger is with the Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, CA 92093-0407. Email: zeger@ucsd.edu .

Notice that $X_{k,1} > 2 - \frac{2}{k+2}$ for all $k \geq 1$, so $f'(X) > 0$ for all $X \geq X_{k,1}$. Also,

$$\begin{aligned} f(X_{k,1}) &= 1 - (1 - 2^{-k-2})^{k+1} \\ &\geq 1 - \left(1 - 2^{-k-2}(k+1) + 2^{-2k-4}\binom{k+1}{2}\right) \\ &= (k+1)2^{-k-2}\left(1 - k2^{-k-3}\right) > 0 \end{aligned}$$

so it must be the case that $f(X) > 0$ for all $X \geq X_{k,1}$. Thus, $\hat{X}_k < X_{k,1}$.

We have

$$\lim_{k\to\infty} (1 - \alpha 2^{-k-2})^{k+1} = 1$$

(to see this, note that the logarithm of the left-hand side approaches 0) so

$$\lim_{k\to\infty} f(X_{k,\alpha}) = 1 - \alpha.$$

Therefore, for all $\alpha > 1$, we have $\lim_{k\to\infty} f(X_{k,\alpha}) < 0$. Thus, for all $\alpha > 1$, there exists a k_α such that for all $k \geq k_\alpha$, we have $\hat{X}_k > X_{k,\alpha}$. In summary, for all $\alpha > 1$ and for all $k \geq k_\alpha$,

$$X_{k,\alpha} < \hat{X}_k < X_{k,1}$$

which implies

$$\log_2 X_{k,\alpha} < E_{0,k} < \log_2 X_{k,1}.$$

Now, for all $\alpha > 0$ and all $k \geq 1$,

$$\log_2 X_{k,\alpha} = 1 + \log_2(1 - \alpha 2^{-k-2}) = 1 - (\log_2 e)\sum_{n=1}^{\infty} \frac{\alpha^n 2^{-n(k+2)}}{n}$$

and thus for all $\alpha > 1$ and all $k \geq 1$,

$$1 \leq \sum_{n=1}^{\infty} \frac{2^{-(n-1)(k+2)}}{n} \leq \frac{1 - E_{0,k}}{(\log_2 e)2^{-k-2}} \leq \sum_{n=1}^{\infty} \frac{\alpha^n 2^{-(n-1)(k+2)}}{n} \leq \sum_{n=1}^{\infty} \alpha^n 2^{-(n-1)(k+2)} = \frac{\alpha}{1 - \alpha 2^{-k-2}}.$$

Thus, for all $\alpha > 1$,

$$1 \leq \limsup_{k\to\infty} \frac{1 - E_{0,k}}{(\log_2 e)2^{-k-2}} \leq \lim_{k\to\infty} \frac{\alpha}{1 - \alpha 2^{-k-2}} = \alpha$$

and therefore

$$\limsup_{k\to\infty} \frac{1 - E_{0,k}}{(\log_2 e)2^{-k-2}} = 1 \leq \liminf_{k\to\infty} \frac{1 - E_{0,k}}{(\log_2 e)2^{-k-2}}$$

which implies

$$\lim_{k\to\infty} \frac{1 - E_{0,k}}{(\log_2 e)2^{-k-2}} = 1,$$

so $E_{0,k}$ converges to 1 at the rate $(\frac{1}{4}\log_2 e)/2^k$. □

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