using the process with hidden states in [3]. The capacity may be estimated to be $H \approx H(10) - H(9) = 0.4682$.

VII. Conclusion

New bounds on the capacity of constrained 2-D codes were derived. The bounds are based on the transfer matrix of superset and subset sources, respectively. The bounds are expressed in terms of capacities of bands and cylinders, which may be determined using well-known one-dimensional results. Two upper and a lower bound applicable to any finite-order constraint were presented. One of the upper bounds was applied to three second-order constraints, improving previous upper bounds. The lower bound was applied to one of these constraints, improving previous results based on transfer matrices.

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Partial Characterization of the Positive Capacity Region of Two-Dimensional Asymmetric Run Length Constrained Channels

Akiko Kato, *Member, IEEE*, and Kenneth Zeger, *Fellow, IEEE*

***Abstract*—A binary sequence satisfies a one-dimensional (d, k) run length constraint if every run of zeros has length at least d and at most k. A two-dimensional binary pattern is (d_1, k_1, d_2, k_2)-constrained if it satisfies the one-dimensional (d_1, k_1) run length constraint horizontally and the one-dimensional (d_2, k_2) run length constraint vertically. For given d_1, k_1, d_2, and k_2, the asymmetric two-dimensional capacity is defined as**

$$C_{d_1,k_1,d_2,k_2} = \lim_{m,n\to\infty} (1/(mn)) \log_2 N_{m,n}^{(d_1,k_1,d_2,k_2)}$$

where $N_{m,n}^{(d_1,k_1,d_2,k_2)}$ denotes the number of (d_1, k_1, d_2, k_2)-constrained $m \times n$ binary patterns. We determine whether the capacity is positive or is zero, for many choices of (d_1, k_1, d_2, k_2).

***Index Terms*—Capacity, constraints, magnetic, optical recording, run length coding.**

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A. Kato, deceased, was with the Department of Mathematical Engineering and Information Physics, Graduate School of Engineering, University of Tokyo, Tokyo 113-8656, Japan.

K. Zeger is with the Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla CA 92093-0407 (e-mail: zeger@ucsd.edu).

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I. Introduction

Run length constraints derive from digital storage applications [7]. For nonnegative integers d and k, a binary sequence is said to satisfy a one-dimensional (d, k)-*constraint* if every run of zeros has length at least d and at most k (if two ones are adjacent in the sequence we say that a run of zeros of length zero is between them). A two-dimensional binary pattern arranged in an $m \times n$ rectangle is said to be (d_1, k_1, d_2, k_2)-*constrained* if it satisfies a one-dimensional (d_1, k_1)-constraint horizontally and a one-dimensional (d_2, k_2)-constraint vertically. The two-dimensional (d_1, k_1, d_2, k_2)-*capacity* is defined as

$$C_{d_1,k_1,d_2,k_2} = \lim_{m,n\to\infty} \frac{\log_2 N_{m,n}^{(d_1,k_1,d_2,k_2)}}{mn}$$

where $N_{m,n}^{(d_1,k_1,d_2,k_2)}$ denotes the number of $m \times n$ rectangles that are (d_1, k_1, d_2, k_2)-constrained. If $d = d_1 = d_2$ and $k = k_1 = k_2$ (this is called the *symmetric constraint*) then the two-dimensional (d_1, k_1, d_2, k_2)-capacity is called the two-dimensional (d, k)-*capacity*, and is denoted by $C_{d,k}$. Two-dimensional run length constraints have recently become a focus of increased study [1], [2], [4]–[7], [9], [15], [16], [21]. A proof was given in [9] that shows the two-dimensional (d, k)-capacities exist, and essentially the same proof shows that the C_{d_1,k_1,d_2,k_2} exist.

The two-dimensional asymmetric *positive capacity region* is the set

$$\{(d_1, k_1, d_2, k_2) : C_{d_1,k_1,d_2,k_2} > 0\}.$$

It is of interest to determine the exact values of the capacities of the various two-dimensional constraints in the positive capacity region, or at least to find good approximations or bounds. A more basic question, however, is to determine which constraints actually lie in the positive capacity region and which do not. We provide here a partial answer to this question.

The exact value of the capacity C_{d_1,k_1,d_2,k_2} has been unknown for all but a few cases. In fact, in all cases when the capacity has been known exactly, its value has been zero and the constraints have been symmetric. The first exactly known two-dimensional capacity was shown in [1] to be $C_{1,2} = 0$ and a complete characterization of which (d, k) integer pairs yield positive capacities for symmetric constraints was given in [9] and is stated as the proposition below.

Proposition 1: $C_{d,k} > 0$ if and only if $k - d \geq 2$ or $(d, k) = (0, 1)$.

Fairly tight upper and lower bounds on the value of $C_{0,1}$ were given in [2], improved in [6], [12], and extended to three-dimensional run length constraints in [12]. In [15], an encoding procedure for the symmetric two-dimensional $(0, 1)$-constrained channel was given whose coding rating comes incredibly close to the capacity $C_{0,1}$. For other positive two-dimensional (d, k)-capacities various bounds were given in [9], [16], and approximations were given in [21]. Asymmetric two-dimensional (d_1, k_1, d_2, k_2)-constraints were studied in [4], which discussed mergings and the Hamming distances between (d_1, k_1, d_2, k_2)-constrained rectangles. Codes for certain other types of constraints in two dimensions were studied in [3], [6], [10], [11], [13], [14], [17]–[21].

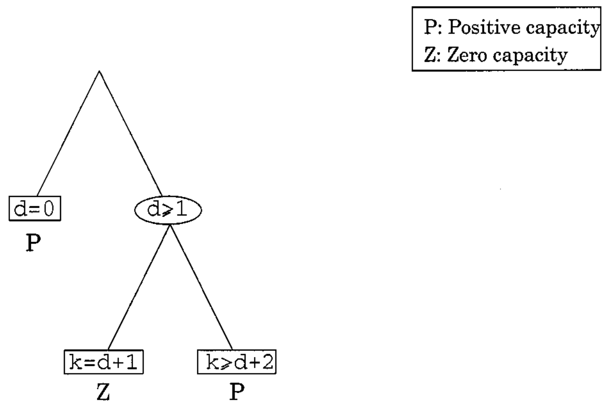

Fig. 1. Zero and positive capacities for the symmetric two-dimensional (d, k) constraints.

In the present correspondence we determine whether or not the two-dimensional capacity is positive, for a large set of asymmetric constraints (d_1, k_1, d_2, k_2). The cases where we determine the asymmetric capacity to be zero (i.e., Theorem 1 part i) and part ii(B)a)) add to the collection of exactly known capacities. We do not, however, determine the exact capacity for any constraints yielding positive capacity. This remains an open problem.

The main results are summarized in Theorem 1. It is interesting to note that for the symmetric constraint (i.e., when $d_1 = d_2$ and $k_1 = k_2$), the capacity is zero whenever d and k are positive and differ by one, whereas for many asymmetric constraints the capacity is positive when the horizontal constraints or the vertical constraints differ by one (e.g., Theorem 1 part ii(B)b)). However, in the asymmetric case if, for example, $k_1 = d_1 + 1 \leq d_2$ then the capacity is zero (by Theorem 1 part i)). The present knowledge of the positive capacity region is shown in Figs. 1 and 2.

The capacity C_{d_1,k_1,d_2,k_2} is symmetric with respect to exchanging horizontal and vertical constraints, and as d_1 or d_2 decreases and k_1 or k_2 increases the (d_1, k_1, d_2, k_2)-constraint allows more valid patterns. Hence, the definition of the (d_1, k_1, d_2, k_2)-constraint implies monotonicities of the capacity in each variable. These facts are stated in the following two lemmas.

Lemma 1: $C_{d_1,k_1,d_2,k_2} = C_{d_2,k_2,d_1,k_1}$ for all d_1, k_1, d_2, and k_2.

Lemma 2: $C_{d_1,k_1,d_2,k_2} \leq C_{\hat{d}_1,\hat{k}_1,\hat{d}_2,\hat{k}_2}$ whenever $\hat{d}_1 \leq d_1 \leq k_1 \leq \hat{k}_1$ and $\hat{d}_2 \leq d_2 \leq k_2 \leq \hat{k}_2$.

The following lemma provides a useful technique for establishing that certain capacities are positive.

Lemma 3: Let A and B be $m \times n$ matrices such that $m \geq k_2$ and $n \geq k_1$. Let AB denote the horizontal concatenation (an $m \times 2n$ matrix) and let A/B denote the vertical concatenation (a $2m \times n$ matrix) of A and B, respectively. If A and B are distinct binary (d_1, k_1, d_2, k_2)-constrained matrices such that $AA, AB, BA, BB, A/A, A/B, B/A$, and B/B all satisfy the (d_1, k_1, d_2, k_2)-constraint, then

$$C_{d_1,k_1,d_2,k_2} \geq \frac{1}{mn}.$$

Proof: Let m' and n' be divisible by m and n, respectively. Any $m' \times n'$ rectangle can be tiled by $m \times n$ rectangles. If A and B satisfy the conditions of the lemma, then the $m' \times n'$ rectangle is (d_1, k_1, d_2, k_2)-constrained whenever each $m \times n$ rectangle is either A or B. Thus $N_{m',n'}^{(d_1,k_1,d_2,k_2)} \geq 2^{(m'/m)(n'/n)}$ and, therefore, $C_{d_1,k_1,d_2,k_2} \geq 1/(mn)$. □

II. Main Results

Theorem 1: Let d_1, k_1, d_2, and k_2 be nonnegative integers such that $d_1 \leq k_1$ and $d_2 \leq k_2$. Let $d = \min(d_1, d_2), D = \max(d_1, d_2)$, $k = \min(k_1, k_2), K = \max(k_1, k_2), \delta = k - D$, and $\Delta = K - d$. Then the following partially characterizes the positive capacity region of two-dimensional run length constrained channels.

i) If $\delta \leq 0$ then $C_{d_1,k_1,d_2,k_2} = 0$.

ii) If $\delta = 1$ then

(A) If $d = 0$ then $C_{d_1,k_1,d_2,k_2} > 0$.

(B) If $d \geq 1$ then

a) If $\Delta \leq 1$ then $C_{d_1,k_1,d_2,k_2} = 0$.

b) If $\Delta > d_1 = d_2$ then $C_{d_1,k_1,d_2,k_2} > 0$.

c) If $\Delta \geq 3$ and $d = 1$ then $C_{d_1,k_1,d_2,k_2} > 0$.

iii) If $\delta \geq 2$ then $C_{d_1,k_1,d_2,k_2} > 0$.

The theorem above reveals whether the capacity is zero or positive for many but not all possible four-tuples (d_1, k_1, d_2, k_2). The only case that is presently not completely characterized in Theorem 1 is part iiB), namely, when $\delta = 1, d \geq 1$, and $\Delta \geq 2$. In part ii(B)b), it is unknown whether the capacity is positive or zero if $d_1 \neq d_2$, for example. If $\delta = 1, d = 1$, and $\Delta = 2$, the only capacities that need be considered are $C_{1,2,1,3}$ and $C_{1,3,2,3}$. But $C_{1,2,1,3} > 0$ from part ii(B)b). Thus if we were able to show that $C_{1,3,2,3} > 0$ then we could replace $\Delta \geq 3$ by $\Delta \geq 2$ in part ii(B)c). However, computer simulation suggests, but does not prove, that perhaps $C_{1,3,2,3} = 0$. This remains an open question.

Proof:

i) Assume without loss of generality that $d_1 \leq d_2$. Then $d_1 \leq k_1 \leq d_2 \leq k_2$ since $\max(d_1, d_2) \geq \min(k_1, k_2)$ from $\delta \leq 0$. Therefore, it suffices to show that $C_{d_1,k_1,d_2,k_2} = 0$ for $d_2 = k_1$ since Lemma 2 implies

$$C_{d_1,k_1,d_2,k_2} \leq C_{d_1,k_1,k_1,k_2}, \qquad d_2 > k_1.$$

Any (d_1, k_1, k_1, k_2)-constrained $(k_1 + 1) \times (k_1 + 1)$ square must have at least one 1 in each row and at most one 1 in each column, and thus must contain exactly one 1 in each row and column. Thus if $n, m > k_1$ then a (d_1, k_1, k_1, k_2)-constrained $m \times n$ rectangle is determined by any $(k_1 + 1) \times (k_1 + 1)$ square in it. Therefore, the number of (d_1, k_1, k_1, k_2)-constrained $m \times n$ rectangles is bounded as $N_{m,n}^{(d_1,k_1,k_1,k_2)} \leq (k_1 + 1)!$ and thus

$$C_{d_1,k_1,k_1,k_2} \leq \lim_{m,n\to\infty} \frac{\log_2 (k_1 + 1)!}{mn} = 0.$$

iiA) Assume without loss of generality $d_1 = 0$, and thus $d_2 = k - \delta = k - 1$. Either $k_1 = k$ and $k_2 = K$ or else $k_1 = K$ and $k_2 = k$. Using Lemma 2, we have

$$C_{0,k,k-1,K} \geq C_{0,k,k-1,k}$$

and

$$C_{0,K,k-1,k} \geq C_{0,k,k-1,k}$$

and thus $C_{d_1,k_1,d_2,k_2} \geq C_{0,k,k-1,k}$. We show that

$$C_{0,k,k-1,k} > 0$$

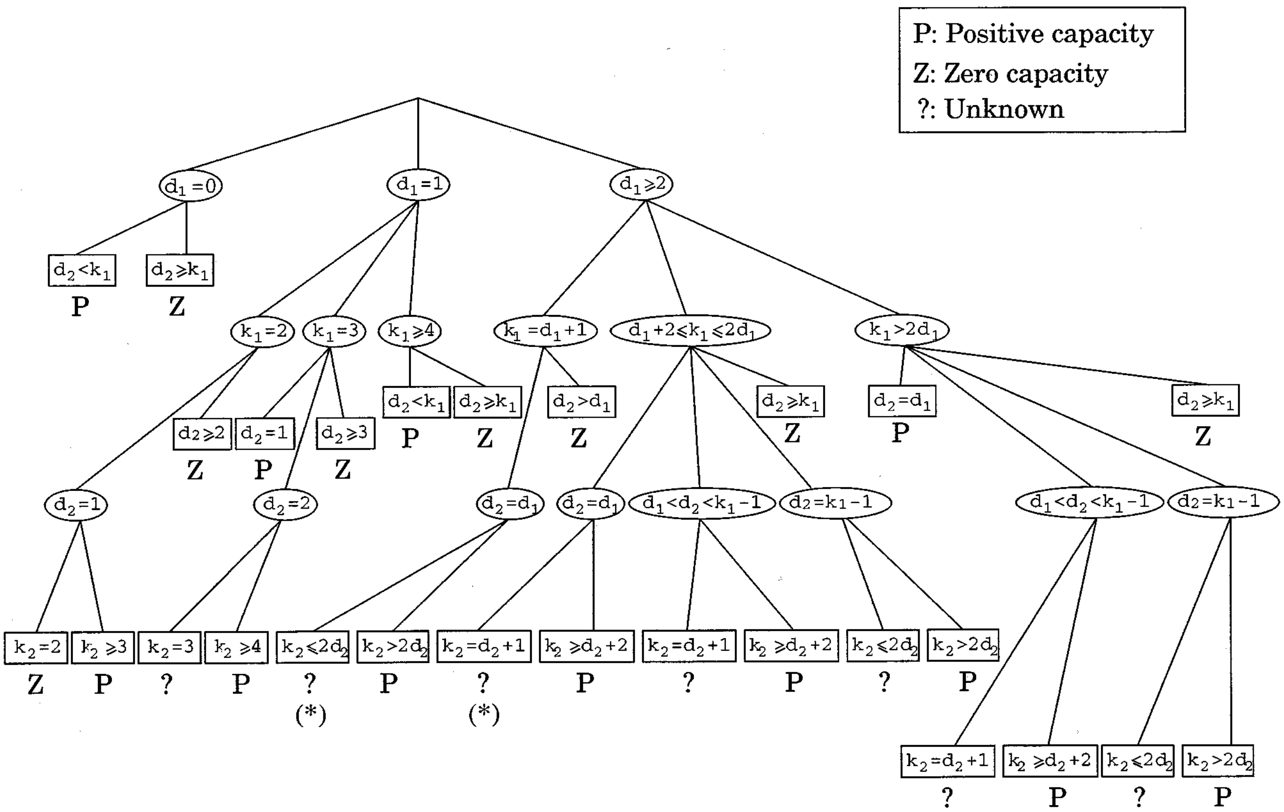

Fig. 2. Zero, positive, and unknown capacities of asymmetric run length constrained channels. These follow from Proposition 1 and Theorem 1. Two unknown capacities indicated by (*) would be zero if Conjecture 1 holds. (We assume $0 \le d_i < k_i \le \infty$ for $i = 1, 2$ and $d_1 \le d_2$.)

for every $k \ge 1$. Let A and B be binary $k(k+1) \times (k+1)$ matrices defined as

$$A = \begin{array}{|c|c|} \hline I_k & \boldsymbol{e}_1 \\ \hline I_k & \boldsymbol{e}_1 \\ \hline \vdots & \vdots \\ \hline I_k & \boldsymbol{e}_1 \\ \hline \vdots & \vdots \\ \hline I_k & \boldsymbol{e}_1 \\ \hline I_k & \boldsymbol{e}_1 \\ \hline \end{array} \qquad B = \begin{array}{|c|c|} \hline I_k & \boldsymbol{e}_1 \\ \hline I_k & \boldsymbol{e}_2 \\ \hline \vdots & \vdots \\ \hline I_k & \boldsymbol{e}_j \\ \hline \vdots & \vdots \\ \hline I_k & \boldsymbol{e}_k \\ \hline I_k & \boldsymbol{0} \\ \hline \end{array}$$

where I_k is the $k \times k$ identity matrix, $\boldsymbol{e}_j$ is a $k \times 1$ column vector in which the jth element from the top is 1 and the other elements are 0, and **0** is the zero column vector. Then the two matrices A and B satisfy the conditions of Lemma 3, and thus for every $k \ge 1$

$$C_{0,k,k-1,k} \ge \frac{1}{k(k+1)^2}.$$

ii(B)a) Since $\delta \le \Delta$ and $\delta = 1$, we have $\Delta = 1$. Together with $d \le D < k \le K$, this implies that $d = D$ and $k = K = d + 1$, and thus $d_1 = d_2 = d < k_1 = k_2 = d + 1$. But $C_{d,d+1} = 0$ for $d \ge 1$ by Proposition 1.

ii(B)b) Since $d = d_1 = d_2$, assume without loss of generality that $k_1 = k$. We have $k = d + 1$ (since $\delta = 1$) and $K > 2d$ (since $\Delta > d$), and, therefore,

$$C_{d_1,k_1,d_2,k_2} = C_{d,d+1,d,K} \ge C_{d,d+1,d,2d+1}$$

(using Lemma 2). Thus it suffices to prove

$$C_{d,d+1,d,2d+1} > 0$$

for $d \ge 1$. Define A and B as the two distinct $3(d+1) \times (2d+3)$ matrices shown in Fig. 3, where I_j is the $j \times j$ identity matrix and 0^j denotes j horizontal or vertical consecutive 0s. Then A and B satisfy the conditions of Lemma 3, giving

$$C_{d,d+1,d,2d+1} \ge \frac{1}{3(d+1)(2d+3)}.$$

ii(B)c) We have $k = \delta + D \ge \delta + d = 1 + 1 = 2$ and $K = \Delta + d \ge 3 + 1 = 4$. Assume without loss of generality $d_1 = d = 1$ and $d_2 = D = k - \delta = k - 1$.

If $k = 2$ then $D = d_2 = 1$ and, therefore,

$$C_{1,K,1,2} = C_{1,2,1,K} \ge C_{1,2,1,3} > 0$$

for $K \ge 4$ by Lemma 2 and part ii(B)b) of this theorem.

If $k = 3$ then either $k_1 = 3$ and $k_2 = K$ or else $k_1 = K$ and $k_2 = 3$. But $C_{1,3,2,K} \ge C_{1,3,2,4}$ and $C_{1,K,2,3} \ge C_{1,4,2,3}$ for $K \ge 4$ by Lemma 2. Therefore, it suffices to show that $C_{1,3,2,4} > 0$ and $C_{1,4,2,3} > 0$. To prove $C_{1,3,2,4} > 0$, define A and B as the binary 21×7 matrices shown in Fig. 4. It can be verified that A and B satisfy the

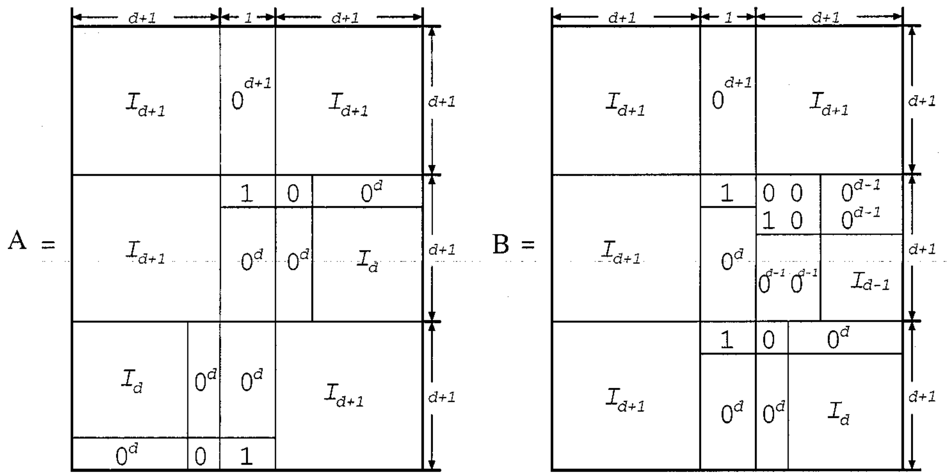

Fig. 3. Two distinct binary $3(d+1) \times (2d+3)$ matrices A and B for the proof of Theorem 1 part ii(B)b) that are $(d, d+1, d, 2d+1)$-constrained for $d \geq 1$. I_j is the $j \times j$ identity matrix and 0^j is j horizontal or vertical consecutive 0s. All tilings of the plane with A and B are also $(d, d+1, d, 2d+1)$-constrained for $d \geq 1$.

conditions of Lemma 3 with $d_1 = 1, k_1 = 3, d_2 = 2$, and $k_2 = 4$. Thus $C_{1,3,2,4} \geq 1/147$. To prove $C_{1,4,2,3} > 0$, define A and B as the binary 16×12 matrices shown in Fig. 5. Then A and B satisfy the conditions of Lemma 3, giving $C_{1,4,2,3} \geq 1/192$.

If $k \geq 4$ then either $k_1 = k$ and $k_2 = K$ or else $k_1 = K$ and $k_2 = k$. Therefore, since $C_{1,k,k-1,K} \geq C_{1,k,k-1,k}$ and $C_{1,K,k-1,k} \geq C_{1,k,k-1,k}$ (by Lemma 2) it suffices to prove

$$C_{1,k,k-1,k} > 0$$

for every $k \geq 4$. Define A and B as the binary $(2k+1) \times (k+1)$ matrices shown in Fig. 6 for $k \geq 4$. The matrices A and B satisfy the conditions of Lemma 3, implying for all $k \geq 4$ that

$$C_{1,k,k-1,k} \geq \frac{1}{(2k+1)(k+1)}.$$

iii) Since $\delta \geq 2$ implies $D \leq k-2$, we have

$$C_{d_1,k_1,d_2,k_2} \geq C_{D,k,D,k} \geq C_{k-2,k,k-2,k} > 0$$

using Lemma 2 for the first two inequalities and Proposition 1 for the third inequality. □

The following corollary states some interesting special cases resulting from Theorem 1.

Corollary 1: Let d_1, k_1, d_2, and k_2 be nonnegative integers such that $d_1 \leq k_1$, and $d_2 \leq k_2$. Let $d = \min(d_1, d_2)$, $D = \max(d_1, d_2)$, $k = \min(k_1, k_2)$, $K = \max(k_1, k_2)$, $\delta = k - D$ and $\Delta = K - d$. Then

i) $C_{d_1,k_1,d_2,k_2} = 0$ whenever $d_1 = k_1$ or $d_2 = k_2$.
ii) $C_{d,d+1,d,2d+1} > 0$ for all $d \geq 0$.
iii) $C_{d_1,k_1,d_2,k_2} > 0$ whenever $D < \min(k, K/2)$.
iv) Let $k_1 > 0$ and $d_2 < k_2$. Then $C_{0,k_1,d_2,k_2} = 0$ if and only if $k_1 \leq d_2$.
v) Let $k_1 \geq 4$ and $d_2 < k_2$. Then $C_{1,k_1,d_2,k_2} = 0$ if and only if $k_1 \leq d_2$.

Proof:

i) This follows from Theorem 1 part i) since $\delta \leq k_1 - d_1$ and $\delta \leq k_2 - d_2$.

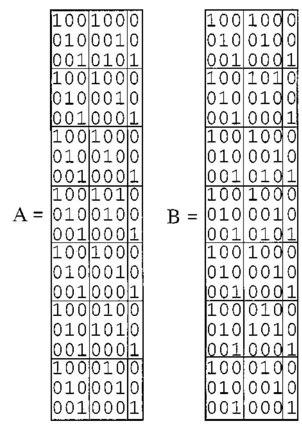

Fig. 4. Two distinct 21×7 binary $(1,3,2,4)$-constrained matrices A and B used to prove Theorem 1 part ii(B)c) with Lemma 3. All tilings of the plane with A and B are also $(1,3,2,4)$-constrained. Note that A and B do not differ below the 12th row nor outside of the 5th and 6th columns.

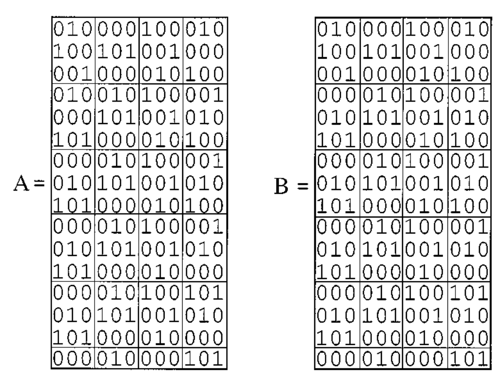

Fig. 5. Two distinct binary 16×12 matrices A and B for the proof of Theorem 1 part ii(B)c) that are $(1,4,2,3)$-constrained. All tilings of the plane with A and B are also $(1,4,2,3)$-constrained. Note that A and B differ only in the two bits located in the fourth and fifth rows of the second column.

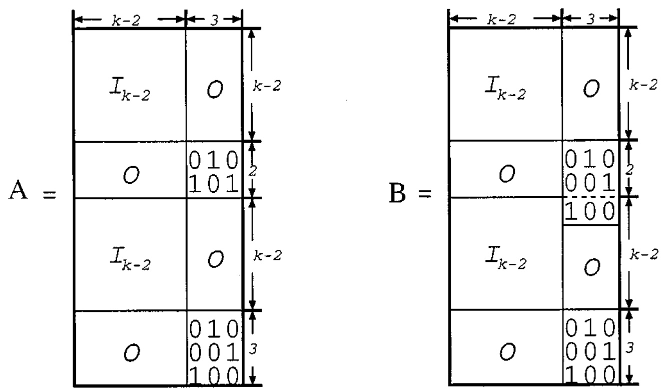

Fig. 6. Two distinct binary $(2k+1) \times (k+1)$ matrices A and B for the proof of Theorem 1 part ii(B)c) for $k \geq 4$ that are $(1, k, k-1, k)$-constrained. I_{k-2} is the $(k-2) \times (k-2)$ identity matrix and O is the zero matrix of appropriate size. All tilings of the plane with A and B are also $(1, k, k-1, k)$-constrained for $k \geq 4$. Note that A and B differ only in two bit locations.

ii) This follows from Proposition 1 for $d = 0$ and from Theorem 1 part ii(B)b) for $d \geq 1$.

iii) If $D < \min(k, K/2)$ then $\delta \geq 1$ and $K > 2D$. If $\delta > 1$ then $C_{d_1,k_1,d_2,k_2} > 0$ by Theorem 1 part iii). If $\delta = 1$ and $d = 0$ then $C_{d_1,k_1,d_2,k_2} > 0$ by Theorem 1 part iiA). So assume $\delta = 1$ and $d \geq 1$.

If $d = D$ then

$$\Delta = K - d > 2D - d = d$$

and thus $C_{d_1,k_1,d_2,k_2} > 0$ by Theorem 1 part ii(B)b).

If $1 = d < D$ then

$$\Delta = K - d > 2D - d \geq 4 - 1 = 3$$

and, therefore, $C_{d_1,k_1,d_2,k_2} > 0$ by Theorem 1 part ii(B)c).

If $2 \leq d < D$ then

$$\Delta - D > \Delta - d = K - 2d > K - 2D > 0$$

which implies $\Delta > D$, and, therefore,

$$C_{d_1,k_1,d_2,k_2} \geq C_{D,k_1,D,k_2} > 0$$

by Lemma 1 and Theorem 1 part ii(B)b).

iv) Taking $d_1 = 0$ in Theorem 1 gives $\min(k_1, k_2) = \delta + d_2$. Therefore, if $k_1 \leq d_2$ then $\delta = k - d_2 \leq k_1 - d_2 \leq 0$. Thus $C_{0,k_1,d_2,k_2} = 0$ by Theorem 1 part i). If $k_1 > d_2$ then $\delta = k - d_2 > 0$. Thus $C_{0,k_1,d_2,k_2} > 0$ by Theorem 1 part iiA) and part iii).

v) The "if" direction follows from part iv) of this corollary and Lemma 2. To establish the "only if" direction suppose that $k_1 > d_2$. Then in Theorem 1, we have $\delta \geq 1, \Delta \geq 3$, and $d = 1$ and, therefore, $C_{1,k_1,d_2,k_2} > 0$ by Theorem 1 part ii(B)c) and part iii). □

III. A Conjecture

We now state a conjecture for which we presently do not have a proof, although computer simulations suggest its plausibility.

Conjecture 1: $C_{d,d+1,d,2d} = 0$ whenever $d \geq 0$.

Conjecture 1 holds for $d = 0$ by Corollary 1 part i) and holds for $d = 1$ by Proposition 1. In contrast, note that $C_{d,d+2,d,2d} > 0$ for every $d \geq 1$. This follows since $C_{1,3,1,2} > 0$ by Theorem 1 part ii(B)b) and since $C_{d,d+2,d,2d} \geq C_{d,d+2} > 0$ for $d \geq 2$ (by Lemma 2 using $2d \geq d + 2$, and by Proposition 1). If Conjecture 1 holds then it would imply that

$$C_{d_1,k_1,d_2,k_2} = 0 \quad \text{whenever } \delta = 1, \ d \geq 0, \ \text{and } \Delta \leq d_1 = d_2.$$

Thus it would characterize with Theorem 1, part ii(B)b) the positive capacity region for $k = d + 1$ and $d_1 = d_2$ as

$$C_{d,K,d,d+1} = C_{d,d+1,d,K} = 0, \qquad \text{if and only if } K \leq 2d.$$

Also, if Conjecture 1 holds then (using Lemma 2) $C_{d_1,k_1,d,d+1} = 0$ whenever $d \leq d_1 \leq k_1 \leq 2d$, and also $C_{d,d+1,d_2,k_2} = 0$ whenever $d \leq d_2 \leq k_2 \leq 2d$.

Figs. 1 and 2 summarize the zero and nonzero capacities given by Proposition 1, Theorem 1, and Corollary 1, assuming $d_1 \leq d_2, d_1 < k_1, d_2 < k_2$.

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