Tradeoff Between Source and Channel Coding

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***Abstract*—A fundamental problem in the transmission of analog information across a noisy discrete channel is the choice of channel code rate that optimally allocates the available transmission rate between lossy source coding and block channel coding. We establish tight bounds on the channel code rate that minimizes the average distortion of a vector quantizer cascaded with a channel coder and a binary-symmetric channel. Analytic expressions are derived in two cases of interest: small bit-error probability and arbitrary source vector dimension; arbitrary bit-error probability and large source vector dimension. We demonstrate that the optimal channel code rate is often substantially smaller than the channel capacity, and obtain a noisy-channel version of the Zador high-resolution distortion formula.**

***Index Terms*—Combined source and channel coding, error exponents, high-resolution vector quantization, separation theorem.**

I. Introduction

SUPPOSE a lossy source coder (vector quantizer) takes an input vector $X \in \boldsymbol{R}^k$ and produces an m-bit output, which is expanded to n bits by a block channel coder and then sent over a binary-symmetric channel. For a fixed transmission rate per source component n/k, what is the best channel code rate m/n to use? This question is studied in this paper, and substantially answered for large n/k in Theorems 1 and 2, which are summarized at the end of this introduction.

An *m-bit vector quantizer* is a function

$$Q_m(x) = \sum_{i=1}^{M} y_{i,m} \mathbf{1}_{S_{i,m}}(x)$$

from $\boldsymbol{R}^k$ to $\{y_{1,m}, \cdots, y_{M,m}\}$, where $M = 2^m$, $\{S_{i,m}\}_{i=1}^M$ is a partition of $\boldsymbol{R}^k$ into disjoint regions or "cells," each of which is represented by a codevector $y_{i,m} \in \boldsymbol{R}^k$ and has indicator function $\mathbf{1}_{S_{i,m}}(\cdot)$. Quantizers are used to compress the information in a random vector X, whose range is usually a continuum, into a discrete set of m-bit indices suitable for transmission over a digital channel. Each index is associated with a single quantizer codevector. For notational simplicity, we abbreviate Q_m as Q, $y_{i,m}$ as y_i, and $S_{i,m}$ as S_i.

Much work has gone into finding quantizers that are optimal for noiseless channels, or what we call *noiseless-optimal*. Such quantizers achieve the distortion $\inf_Q D_m(Q)$, where

$$D_m(Q) = \sum_{i=1}^{M} \int_{S_i} \|x - y_i\|^p f(x)\, dx,$$

$\|\cdot\|^p$ represents the pth power of the usual Euclidean l_2 norm, and f is the probability density function of X. Let Q^* denote a quantizer achieving the infimum. Then Q^* is known to obey nearest neighbor and, for $p = 2$, centroid conditions [10], [11], which sometimes determine Q^* uniquely [13]. Many other properties are known, such as the quantizer's limiting (in M) "point density" [6].

When the channel is binary-symmetric (BSC), the m-bit ith quantizer index changes into the jth index with probability $\epsilon^{d_{i,j}}(1-\epsilon)^{m-d_{i,j}}$, where $d_{i,j}$ is the Hamming distance between the ith and jth quantizer indices. The distortion of using Q then becomes

$$D_m(Q,\epsilon) \overset{\text{def}}{=} \sum_{i,j=1}^{M} \epsilon^{d_{i,j}}(1-\epsilon)^{m-d_{i,j}} \int_{S_i} \|x-y_j\|^p f(x)\, dx.$$

Quantizers that minimize this distortion are known to obey generalized nearest neighbor and centroid conditions [3], [8], [9], but very little else is known about the complicated structure of, or even how to find, a Q that minimizes $D_m(Q,\epsilon)$. It is therefore popular, in practice, to adapt quantizers that work well over noiseless channels (such as a noiseless-optimal Q^*) for use over noisy channels. This paper studies the cascading of a quantizer with a channel coder to minimize the average end-to-end distortion over a BSC.

Channel coding reduces the codeword error probability by adding redundancy to the binary representation of the quantizer indices. The m-bit index produced by the quantizer is expanded to an n-bit channel codeword, where the extra $n-m$ bits are used to guard against channel errors. We address the question: For a given channel transmission rate per source component $R = n/k$ (dictated, say, by available channel bandwidth), what channel code rate $r = m/n$ minimizes the average distortion from sender to receiver?

For a real-valued source with distortion-rate function $D(\cdot)$ and a noisy binary channel of capacity C, Shannon's well-known "separation theorem" says that one can transmit R bits per source sample across the channel and achieve a distortion arbitrarily close to $D(CR)$ by independently designing source and channel coders. However, this requires long blocks of source symbols $(k \to \infty)$, and long channel codewords $(n \to \infty)$. In practice, the vector dimension k is often bounded

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because of delay constraints. It is therefore generally of greater interest to determine the best achievable performance of a cascaded source and channel coder for a *fixed* value of k. In particular, a fundamental open problem is to determine the optimal channel code rate r, as a function of k. We solve this problem, in part, by obtaining tight upper and lower bounds on the optimal r using high-resolution quantization theory. These bounds turn out to be independent of the source distribution and of the transmission rate, as $R \to \infty$.

There is a clear tradeoff between channel coding redundancy versus source coding resolution. Low redundancy (large r) can be had only at the expense of high probability of codeword error. In this case, the channel codeword bits are used almost exclusively for carrying quantization information, and there is little channel error correction. The source coding or quantization distortion is small, but since the channel decoding error probability is relatively high, the sender-to-receiver distortion may also be unacceptably high. The limiting case of zero redundancy $(r = 1)$ generally has strict performance limits that cannot be exceeded. For example, in the case of a uniformly distributed scalar source, the mean-squared distortion of using Q^* is bounded from below by some known function of ϵ, no matter how large R is [2].

On the other hand, high redundancy (small r) occurs at the expense of low source coding resolution. In this case, the channel decoding error probability is small, but the source coding distortion is relatively high, thus again possibly yielding a large total distortion. The limiting case of $r = 0$ conveys no information about the source at all.

Between these two extremes there exists a channel code rate r that minimizes distortion. We examine the optimal choice of r in the high-resolution limit (when R is large). Analytic results are given in two cases: low channel bit-error probability and arbitrary source vector dimension; arbitrary channel bit-error probability and large source vector dimension. The minimizing r is generally a function of the bit-error probability ϵ, since as ϵ decreases, less redundancy is required to achieve good performance.

High-resolution quantization theory has a long history, is relatively well-understood, and is essentially the only known technique for obtaining analytic expressions for quantizer performance. Furthermore, the high-resolution theory is known often to model accurately even low-resolution quantizers. In many low-resolution source coding schemes, high-resolution quantizers act as key embedded building blocks in the overall compression system. For these reasons, high-resolution theory is a useful tool.

In [15] it is shown that the distortion decays exponentially with the channel transmission rate R, on a BSC, but no explicit rate of decay is identified. We show, in Theorem 1 that, for small bit-error probability ϵ, the lowest pth-power distortion, averaged over all codevector-to-codeword assignments over a BSC, is achieved for some channel code rate r satisfying

$$1 - \frac{2p}{k}\frac{\log\log 1/\epsilon}{\log 1/\epsilon} + O\left(\frac{1}{\log 1/\epsilon}\right) + o(1) \le r \le 1 - \frac{p}{k}\frac{\log\log 1/\epsilon}{\log 1/\epsilon} + O\left(\frac{1}{\log 1/\epsilon}\right) + o(1).$$

The $O(1/\log 1/\epsilon)$ terms go to zero as $\epsilon \to 0$, uniformly in R; the $o(1)$ terms go to zero as $R \to \infty$. This result is valid for any fixed k, and is consistent with the fact that $r \to 1$ as $\epsilon \to 0$ for any R.

On the other hand, in Theorem 2 we show that, for large k, the channel code rate that minimizes average distortion is

$$r = C - \frac{\alpha}{\sqrt{k}} + O\left(\frac{1}{k}\right) + o(1)$$

where $C = 1 + \epsilon\log\epsilon + (1-\epsilon)\log(1-\epsilon)$ is the BSC capacity and α is a function of p and ϵ. The $O(1/k)$ term goes to zero as $k \to \infty$, uniformly in R, and the $o(1)$ term goes to zero as $R \to \infty$. Note that, unlike Theorem 1, this is an asymptotic equality and is valid for any fixed ϵ. Furthermore, this equality is consistent with the fact that $r \to C$ as $k \to \infty$ for any R (via Shannon's rate distortion and channel coding theorems).

The corresponding distortion in Theorems 1 and 2 is

$$2^{-pRr+O(1)}$$

as $R \to \infty$, thus yielding a noisy-channel version of the well-known Zador [14] distortion $2^{-pR+O(1)}$. The optimal channel code rate $r \in (0,1)$ is the penalty imposed on source coding resolution due to channel noise, and is always below the channel capacity C. The conclusions of both theorems are especially appealing because they are independent of the source density f.

Section II lists the assumptions used throughout the paper. Sections III and IV contain the analytical results, including Theorems 1 and 2. Section V provides numerical illustrations of the analytical results. Proofs of lemmas needed along the way are relegated to appendices. Some items of notation: i) Let $b_1, b_2, \cdots$ be a sequence of positive numbers. We say that $a_i = O(b_i)$ as $i \to \infty$ if $|a_i|/b_i \le c$ for some $c > 0$ and all i sufficiently large. We say that $a_i = \Omega(b_i)$ if $|a_i|/b_i \ge c$ for some $c > 0$ and all i sufficiently large. Finally, $a_i = o(b_i)$ if $\lim_{i\to\infty} a_i/b_i = 0$. ii) Unless specified otherwise, logarithms are base two.

II. Assumptions

In the system we consider, a transmitter, consisting of a source encoder (vector quantizer) followed by a channel encoder, sends its output over a BSC to a receiver consisting of a channel decoder and a source decoder. We call such a system a *cascaded vector quantizer and channel coder*, and now describe what is assumed about the source and channel encoders/decoders.

It is assumed that the random vector X being quantized has probability density function f supported in K, a closed bounded subset of $\boldsymbol{R}^k$ with nonempty interior. We assume that the source encoder/decoder pair uses the nearest neighbor rule to partition K and achieves the (noiseless channel) distortion

$$D_m(Q) = 2^{-pm/k+O(1)} = 2^{-pRr+O(1)} \tag{1}$$

as $R \to \infty$. We call any vector quantizer that achieves this exponential rate of decay with R a *good vector quantizer*. It is well known [1], [14], that the noiseless-optimal quantizer Q^* is good, as are many suboptimal, including the uniform and

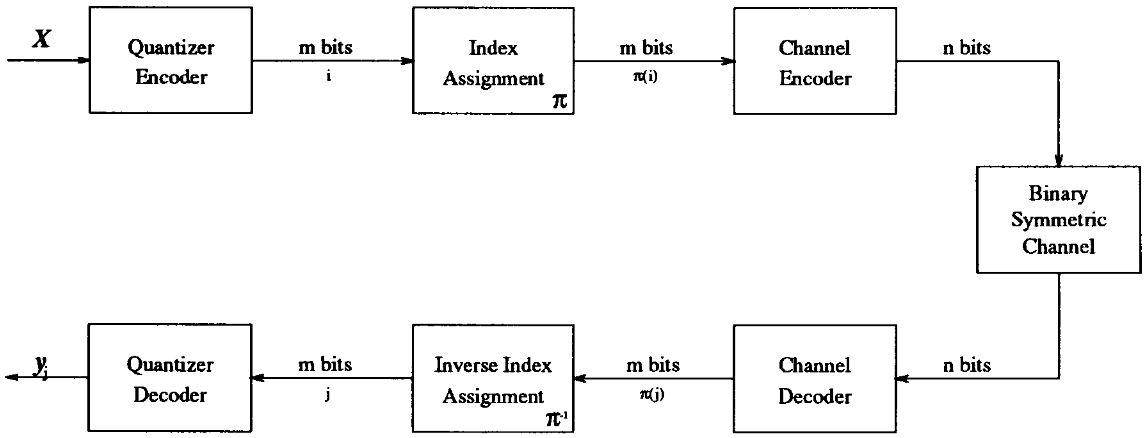

Fig. 1. Cascaded vector quantizer and channel coder system.

other lattice-based, quantizers. Besides depending on the type of quantizer, the $O(1)$ term generally depends on f, k, and p. The quantizer resolution is Rr bits per source component.

It is assumed that the channel encoder employs $n - m$ bits of redundancy added to the original m-bit quantization index. Furthermore, the channel decoder is maximum-likelihood. The channel codeword size is therefore n bits, and the number of channel codeword bits per source component is n/k.

We also assume that the quantizer codevectors $y_1, \cdots, y_M$ are mapped to $M = 2^m = 2^{kRr}$ n-bit channel codewords through a permutation mapping π, commonly called an *index assignment*, which is an element of the symmetric group on M letters. That is, y_i gets mapped to the $\pi(i)$th codeword (using any convenient ordering of the codewords). Fig. 1 shows a system block diagram. The choice of π is assumed random and equally likely from the $M!$ different possible index assignments, although we do not use this particular assumption until Section III-B.

Throughout the paper, we consider only r below the channel capacity C, since, for $r > C$, it can be shown (see Section III-B) that the average distortion is bounded away from zero, no matter how large R is. The next section finds analytic bounds on the optimal channel code rate r, when the channel bit-error probability is small and the channel transmission rate R is large.

III. Fixed Source Vector Dimension and Small Bit-Error Probability

Theorem 1: The minimum pth-power distortion, averaged over all index assignments, of a k-dimensional cascaded good vector quantizer and channel coder that transmits over a binary-symmetric channel with bit-error probability ϵ, is achieved with a channel code rate r satisfying

$$\begin{aligned}1 - \frac{2p}{k}\frac{\log\log 1/\epsilon}{\log 1/\epsilon} + O\left(\frac{1}{\log 1/\epsilon}\right) + o(1) \\ \le r \le 1 - \frac{p}{k}\frac{\log\log 1/\epsilon}{\log 1/\epsilon} + O\left(\frac{1}{\log 1/\epsilon}\right) + o(1)\end{aligned}$$

where the $O(1/\log 1/\epsilon)$ terms go to zero as $\epsilon \to 0$, uniformly in the channel transmission rate R, and the $o(1)$ terms go to zero as $R \to \infty$. These bounds are independent of the density of the source vector.

The proof is deferred until later, and follows from the upper and lower bounds on average distortion developed in Sections III-A and III-B.

Define $q(j|i)$ as the probability that the channel decoder decides that the jth channel codeword was sent when, in fact, the ith was sent. Then the cascaded quantizer/channel-coder system with a BSC has distortion

$$D_R(Q, \epsilon, \pi) \stackrel{\text{def}}{=} \sum_{i,j=1}^{M} q(\pi(j)|\pi(i)) \int_{S_i} \|x - y_j\|^p f(x)\, dx.$$

A. Distortion Upper Bound

The distortion may be written

$$\begin{aligned}D_R(Q, \epsilon, \pi) &= \sum_{i=1}^{M} q(\pi(i)|\pi(i)) \int_{S_i} \|x - y_i\|^p f(x)\, dx \\ &\quad + \sum_{\substack{i,j=1\\ i\ne j}}^{M} q(\pi(j)|\pi(i)) \int_{S_i} \|x - y_j\|^p f(x)\, dx \\ &\le \sum_{i=1}^{M} \int_{S_i} \|x - y_i\|^p f(x)\, dx \\ &\quad + O(1) \sum_{i=1}^{M} P(S_i) \sum_{\substack{j=1\\ j\ne i}}^{M} q(\pi(j)|\pi(i)) \\ &= D_m(Q) + O(1) \cdot \sum_{i=1}^{M} P(S_i) P_{e|\pi(i)} \\ &\le D_m(Q) + O(1) \cdot \max_{1\le i\le M} P_{e|\pi(i)}\end{aligned}$$

as $R \to \infty$, where $P_{e|\pi(i)}$ is the probability of incorrect decoding, given that the $\pi(i)$th codeword is sent. The $O(1)$ term is positive and is due to the fact that f has support K, and y_j is contained in K for all j.

Shannon's channel coding theorem guarantees that, for channel code rates r below capacity, channel codes exist for which

$$\max_{1\le i\le M} P_{e|i} \le 2^{-nE_{\rm ex}(r)+o(n)} = 2^{-kRE_{\rm ex}(r)+o(R)} \quad (2)$$

as $R \to \infty$, where

$$E_{\rm ex}(r) = \sup_{\rho\ge 1}\{\rho[1-r-\log(1+\delta^{1/\rho})]\} \quad (3)$$

is the "expurgated error exponent" for the binary-symmetric channel [4], [7], and $\delta = 2\sqrt{\epsilon(1-\epsilon)}$. Hence

$$D_R(Q,\epsilon,\pi) \le 2^{-pRr+O(1)} + 2^{-kRE_{\rm ex}(r)+o(R)}. \quad (4)$$

Note that this upper bound does not depend on the index assignment π, or the source density f. Since $E_{\rm ex}(r)$ is known to be a convex decreasing function of r, the right-hand side of (4) is minimized for large R by choosing r so that the exponents in both terms are within $O(1)$ of each other (see also [15]). For otherwise, the term whose exponent is less negative would dominate the sum when R is sufficiently large. Let $r_{\rm ex}$ be the resulting channel code rate; then $r_{\rm ex}$ obeys

$$E_{\rm ex}(r_{\rm ex}) = (p/k)r_{\rm ex} + o(1) \quad (5)$$

as $R \to \infty$. Since $E_{\rm ex}(r) > 0$ for $r < C$ and $E_{\rm ex}(r) = 0$ for $r \ge C$, where

$$C = 1 + \epsilon\ \log\epsilon + (1-\epsilon)\ \log(1-\epsilon)$$

is the channel capacity, it follows that $r_{\rm ex} \in (0, C)$ for R sufficiently large. The next lemma provides an analytic solution to (5) that is accurate when the bit-error probability ϵ is small.

Lemma 1: For any p and k, suppose that c_δ satisfies

$$\frac{p}{k}2^{c_\delta} - \frac{(p/k)(\log\log 1/\delta + \log e + c_\delta) - 2^{-c_\delta}}{\log 1/\delta} - 1 = 0 \quad (6)$$

and is bounded as $\delta \to 0$. Then, the channel code rate that minimizes the upper bound on distortion in (4) is

$$r_{\rm ex} = 1 - 2^{-c_\delta}\left(\frac{\log\log 1/\delta + \log e + c_\delta}{\log 1/\delta}\right) + O\left(\frac{\log\log 1/\delta}{\log^2 1/\delta}\right) + o(1), \quad (7)$$

where the $O((\log\log 1/\delta)/\log^2 1/\delta)$ term goes to zero as $\delta \to 0$, uniformly in the channel transmission rate R, and the $o(1)$ term goes to zero as $R \to \infty$. This result is independent of the index assignment π.

Proof: See Appendix I.

Remarks: Equation (7) is equivalent to

$$\limsup_{\delta\to 0}\ \lim_{R\to\infty} \frac{\log^2 1/\delta}{\log\log 1/\delta} \cdot \left| r_{\rm ex}(\delta,R) - 1 + 2^{-c_\delta}\left(\frac{\log\log 1/\delta + \log e + c_\delta}{\log 1/\delta}\right)\right| < \infty.$$

The left-hand side of (6) is continuous in c_δ and is negative at $c_\delta = \log(k/p) - 1$ and positive at $c_\delta = \log(k/p) + 1$ for sufficiently small δ. Thus there exists a c_δ satisfying (6) that is bounded as $\delta \to 0$. We conclude that the left-hand side of (6) is $(p/k)2^{c_\delta} - 1 + O((\log\log 1/\delta)/\log 1/\delta)$ as $\delta \to 0$, and hence $c_\delta \to \log k/p$.

One important conclusion to draw from this lemma is that for small ϵ

$$\begin{aligned} r_{\rm ex} &= 1 - \frac{p}{k}\frac{\log\log 1/\delta}{\log 1/\delta} + O\left(\frac{1}{\log 1/\delta}\right) + o(1) \\ &= 1 - \frac{2p}{k}\frac{\log\log 1/\epsilon}{\log 1/\epsilon} + O\left(\frac{1}{\log 1/\epsilon}\right) + o(1). \end{aligned} \quad (8)$$

Combining this with (4), we see that there exist channel codes such that

$$\begin{aligned} D_R(Q,\epsilon,\pi) &\le 2^{-pRr_{\rm ex}+O(1)} \\ &= 2^{-pR[1-(2p/k)(\log\log 1/\epsilon)/\log 1/\epsilon + O(1/\log 1/\epsilon) + o(1)]}. \end{aligned}$$

The right-hand side clearly does not depend on π or f.

B. Distortion Lower Bound

We derive a lower bound on average distortion and minimize it over all channel code rates $r \in (0,1)$ when the channel transmission rate R is large. The lower bound explicitly uses the assumption that the index assignment π is chosen randomly and equally likely from the $M!$ different possible assignments. Before we can proceed, a lemma is needed.

Lemma 2: Any good vector quantizer with codevectors $y_1, \cdots, y_M \in K$ satisfies

$$\min_{x\in K}\sum_{i=1}^{M} \|x - y_i\|^p = \Omega(M)$$

as $M \to \infty$.

Proof: See Appendix II.

The distortion, averaged over all index assignments, is

$$\begin{aligned} &\overline{D}_R(Q,\epsilon) \\ &\stackrel{\rm def}{=} \frac{1}{M!}\sum_\pi D_R(Q,\epsilon,\pi) \\ &= \frac{1}{M!}\sum_\pi \sum_{i,j=1}^{M} q(\pi(j)|\pi(i)) \int_{S_i} \|x-y_j\|^p f(x)\,dx \\ &= \frac{1}{M!}\sum_{i=1}^{M}\sum_\pi q(\pi(i)|\pi(i)) \int_{S_i} \|x-y_i\|^p f(x)\,dx \\ &\quad + \frac{1}{M!}\sum_{\substack{i,j=1\\ i\ne j}}^{M}\sum_\pi q(\pi(j)|\pi(i)) \int_{S_i} \|x-y_j\|^p f(x)\,dx \end{aligned} \quad (9)$$

$$\begin{aligned} &= \frac{1}{M}\sum_{i=1}^{M} q(i|i) \sum_{j=1}^{M} \int_{S_j} \|x-y_j\|^p f(x)\,dx \\ &\quad + \frac{1}{M!}\sum_{\substack{i,j=1\\ i\ne j}}^{M}\sum_\pi q(\pi(j)|\pi(i)) \int_{S_i} \|x-y_j\|^p f(x)\,dx \end{aligned} \quad (10)$$

$$
\begin{aligned}
&= 2^{-pRr+O(1)} \frac{1}{M} \sum_{i=1}^{M} (1 - P_{e|i}) \\
&\quad + \frac{1}{M!}(M-2)! \sum_{\substack{k,l=1 \\ k \neq l}}^{M} q(l|k) \\
&\quad \cdot \sum_{i=1}^{M} \int_{S_i} \sum_{\substack{j=1 \\ j \neq i}}^{M} ||x - y_j||^p f(x)\, dx \\
&\geq 2^{-pRr+O(1)}(1 - P_e) \\
&\quad + \frac{1}{M(M-1)} \sum_{\substack{k,l=1 \\ k \neq l}}^{M} q(l|k) \\
&\quad \cdot \sum_{i=1}^{M} P(S_i) \min_{x \in K} \sum_{\substack{j=1 \\ j \neq i}}^{M} ||x - y_j||^p \\
&= 2^{-pRr+O(1)}(1 - P_e) \\
&\quad + \frac{1}{M(M-1)} \Omega(M) \sum_{\substack{k,l=1 \\ k \neq l}}^{M} q(l|k) \qquad (11) \\
&= 2^{-pRr+O(1)}(1 - P_e) + \Omega(1) \frac{1}{M} \sum_{k=1}^{M} P_{e|k} \\
&= 2^{-pRr+O(1)}(1 - P_e) + \Omega(1) P_e \qquad (12)
\end{aligned}
$$

where $P_e = (1/M)\Sigma_{i=1}^{M} P_{e|i}$ is the average probability of error for the channel decoder, and Lemma 2 is used in (11), yielding a positive $\Omega(1)$ term in (12). Clearly, if P_e does not decay to zero as $R \to \infty$, then the right-hand side of (12) also does not decay to zero. To minimize the right-hand side of (12) for large R, we therefore consider only r below the channel capacity, and only channel codes for which $P_e \to 0$.

A lower bound on P_e appears in [12, Theorem 2], where it is shown that

$$P_e \geq 2^{-nE_{\text{sp}}(r)+o(n)} = 2^{-kRE_{\text{sp}}(r)+o(R)}$$

where

$$
\begin{aligned}
E_{\text{sp}}(r) = \sup_{\rho \geq 0} \{&\rho(1 - r) \\
&- (\rho + 1)\log[\epsilon^{1/(1+\rho)} + (1 - \epsilon)^{1/(1+\rho)}]\} \qquad (13)
\end{aligned}
$$

is the "sphere-packing exponent" of the binary-symmetric channel. It follows that

$$\overline{D}_R(Q, \epsilon) \geq 2^{-pRr+O(1)} + 2^{-kRE_{\text{sp}}(r)+o(R)}. \qquad (14)$$

This bound does not depend on the source density f. To get a lower bound that does not depend on r, we minimize the right-hand side of (14) over r in the same way that we minimize (4) for large R. Let r_{sp} be the resulting rate. Then

$$E_{\text{sp}}(r_{\text{sp}}) = (p/k)r_{\text{sp}} + o(1) \qquad (15)$$

as $R \to \infty$. For R sufficiently large, $r_{\text{sp}} \in (0, C)$. The next lemma provides an analytic solution to (15) that is accurate when ϵ is small.

Lemma 3: For any p and k, let c_ϵ be any solution to

$$\frac{p}{k} 2^{c_\epsilon} + \frac{(1 - p/k)(\log \log 1/\epsilon + \log e + c_\epsilon) + 2^{-c_\epsilon}}{\log 1/\epsilon} - 1 = 0 \qquad (16)$$

that is bounded as $\epsilon \to 0$. Then, the channel code rate that minimizes the lower bound on distortion is

$$
\begin{aligned}
r_{\text{sp}} = 1 - 2^{-c_\epsilon} &\left(\frac{\log \log 1/\epsilon + \log e + c_\epsilon}{\log 1/\epsilon} \right) \\
&+ O\left(\frac{\log \log 1/\epsilon}{\log^2 1/\epsilon} \right) + o(1) \qquad (17)
\end{aligned}
$$

where the $O((\log \log 1/\epsilon)/\log^2 1/\epsilon)$ term goes to zero as the bit-error probability $\epsilon \to 0$, uniformly in the channel transmission rate R, and the $o(1)$ term goes to zero as $R \to \infty$.

Proof: See Appendix III.

Remark: A simple argument, identical to the one contained in the remarks following Lemma 1, shows that there exists a c_ϵ satisfying (16) such that $c_\epsilon \to \log k/p$ as $\epsilon \to 0$.

The expansions of r_{ex} and r_{sp} given in (7) and (17) are remarkably similar. We can conclude from Lemma 3 that, for small ϵ

$$r_{\text{sp}} = 1 - \frac{p}{k} \frac{\log \log 1/\epsilon}{\log 1/\epsilon} + O\left(\frac{1}{\log 1/\epsilon} \right) + o(1) \qquad (18)$$

which differs from the expression for r_{ex} given in (8) by only a factor of two in the second term. Combining (18) with (14) yields

$$
\begin{aligned}
\overline{D}_R(Q, \epsilon) &\geq 2^{-pRr_{\text{sp}}+O(1)} \\
&= 2^{-pR[1-(p/k)(\log\log 1/\epsilon)/\log 1/\epsilon + O(1/\log 1/\epsilon) + o(1)]}. \qquad (19)
\end{aligned}
$$

Proof of Theorem 1: Because the upper bound in (4) does not depend on π, it holds that

$$\overline{D}_R(Q, \epsilon) \leq 2^{-pRr+O(1)} + 2^{-kRE_{\text{ex}}(r)+o(R)}.$$

Comparing this inequality with (14), we see that the r that minimizes the average distortion obeys

$$r_{\text{ex}} \leq r \leq r_{\text{sp}} \qquad (20)$$

when R is sufficiently large. Lemmas 1 and 3 now complete the proof of Theorem 1. □

Remarks on Theorem 1: i) Theorem 1 implies that, for small ϵ, somewhere between at least approximately $(p/k) \cdot (\log \log 1/\epsilon)/\log 1/\epsilon$ fraction, and at most approximately $(2p/k)(\log \log 1/\epsilon)/\log 1/\epsilon$ fraction, of the transmission rate R should be used for channel coding to minimize the average distortion. Clearly, these upper and lower bounds are quite close to each other. For the optimal r, the corresponding average distortion obeys

$$\overline{D}_R(Q, \epsilon) = 2^{-pRr+O(1)}$$

independently of the probability density function of the source vector.

ii) The optimal code rate $r \in (0, 1)$ can be thought of as a penalty due to channel noise, since the Zador (noiseless channel) distortion [14] corresponds to $r = 1$. The statement of Theorem 1 is consistent with the fact that $r \to 1$ as $\epsilon \to 0$ for any R.

iii) The small ϵ bounds on r given in Theorem 1 are, of course, approximations of $r_{\rm ex}$ and $r_{\rm sp}$. Bounds suitable for any ϵ can be obtained by solving (5) and (15) numerically, or by using the additional terms in the asymptotic expansions given in Lemmas 1 and 3 (see also Section V).

IV. Fixed Bit Error Probability and Large Source Vector Dimension

The following theorem shows that when the source dimension exceeds a certain threshold, the optimal channel code rate can be exactly characterized since the code rate upper and lower bounds coincide.

Theorem 2: Let a source vector have dimension

$$k > p(\beta - \beta \log \beta - \gamma)/(\gamma - \beta \log \beta)$$

where

$$\beta = \sqrt{\epsilon} + \sqrt{1-\epsilon}$$

and

$$\gamma = -\sqrt{\epsilon} \log \sqrt{\epsilon} - \sqrt{1-\epsilon} \log \sqrt{1-\epsilon}.$$

Then, the minimum pth-power distortion, averaged over all index assignments, of a k-dimensional cascaded good vector quantizer and channel coder that transmits over a binary-symmetric channel with bit-error probability ϵ is achieved when the channel code rate is

$$r = C - \frac{\alpha}{\sqrt{k}} + O\left(\frac{1}{k}\right) + o(1) \tag{21}$$

where

$$\alpha = \left[(2pC/\log e)[\epsilon \log^2 \epsilon + (1-\epsilon) \log^2(1-\epsilon) - (1-C)^2]\right]^{1/2}$$

and

$$C = 1 + \epsilon \log \epsilon + (1-\epsilon) \log(1-\epsilon)$$

is the channel capacity. The $O(1/k)$ term goes to zero as $k \to \infty$, uniformly in the channel transmission rate R, and the $o(1)$ term goes to zero as $R \to \infty$.

The proof is deferred until later, and follows from the upper and lower bounds on average distortion developed in Sections IV-A and IV-B.

A. Distortion Upper Bound

We replace the upper bound on probability of error given in (2) with

$$\max_{1 \le i \le M} P_{e|i} \le 2^{-kRE_{\rm r}(r)+2} \tag{22}$$

where

$$E_{\rm r}(r) = \max_{0 \le \rho \le 1} \{\rho(1-r) - (\rho+1) \log[\epsilon^{1/(1+\rho)} + (1-\epsilon)^{1/(1+\rho)}]\} \tag{23}$$

is the "random coding exponent" for the binary-symmetric channel [4]. Observe that $E_{\rm r}$ in (23) and $E_{\rm sp}$ in (13) differ only in the range of ρ over which maximization is performed.

The same argument used in Section III now yields

$$D_R(Q, \epsilon, \pi) \le 2^{-pRr+O(1)} + 2^{-kRE_{\rm r}(r)+O(1)} \tag{24}$$

and the channel code rate $r_{\rm ra}$ that minimizes this upper bound satisfies

$$E_{\rm r}(r_{\rm ra}) = (p/k)r_{\rm ra} + O(1/R) \tag{25}$$

as $R \to \infty$. Note that this upper bound does not depend on the index assignment π, or the source density f. The next lemma provides an analytic solution to (25) that is accurate when k is large.

Lemma 4: For any fixed bit-error probability ϵ, the channel code rate that minimizes the upper bound on distortion in (24) is

$$r_{\rm ra} = C - \frac{\alpha}{\sqrt{k}} + O\left(\frac{1}{k}\right) + O\left(\frac{1}{R}\right) \tag{26}$$

where

$$\alpha = \left[(2pC/\log e) \cdot [\epsilon \log^2 \epsilon + (1-\epsilon) \log^2(1-\epsilon) - (1-C)^2]\right]^{1/2}$$

and

$$C = 1 + \epsilon \log \epsilon + (1-\epsilon) \log(1-\epsilon)$$

is the binary-symmetric channel (BSC) capacity. The $O(1/k)$ term goes to zero as the source vector dimension $k \to \infty$, uniformly in the channel transmission rate R, and the $O(1/R)$ term goes to zero as $R \to \infty$. This result is independent of the index assignment π, and the source density f.

Proof: See Appendix IV.

Combining Lemma 4 with (24) yields

$$D_R(Q, \epsilon, \pi) \le 2^{-pRr_{\rm ra}+O(1)} = 2^{-pR[C-\alpha/\sqrt{k}+O(1/k)]+O(1)}.$$

B. Distortion Lower Bound

From (19)

$$\overline{D}_R(Q, \epsilon) \ge 2^{-pRr_{\rm sp}+O(1)}$$

where $r_{\rm sp}$ is the solution to $E_{\rm sp}(r_{\rm sp}) = (p/k)r_{\rm sp} + o(1)$. The following lemma shows that $r_{\rm ra}$ and $r_{\rm sp}$ meet as $R \to \infty$, provided that k is large enough.

Lemma 5: Let

$$\beta = \sqrt{\epsilon} + \sqrt{1-\epsilon}$$

and

$$\gamma = -\sqrt{\epsilon} \log \sqrt{\epsilon} - \sqrt{1-\epsilon} \log \sqrt{1-\epsilon}.$$

If

$$k > \frac{p(\beta - \beta \log \beta - \gamma)}{\gamma - \beta \log \beta} \tag{27}$$

then

$$r_{\rm sp} = r_{\rm ra} + o(1)$$

as $R \to \infty$.

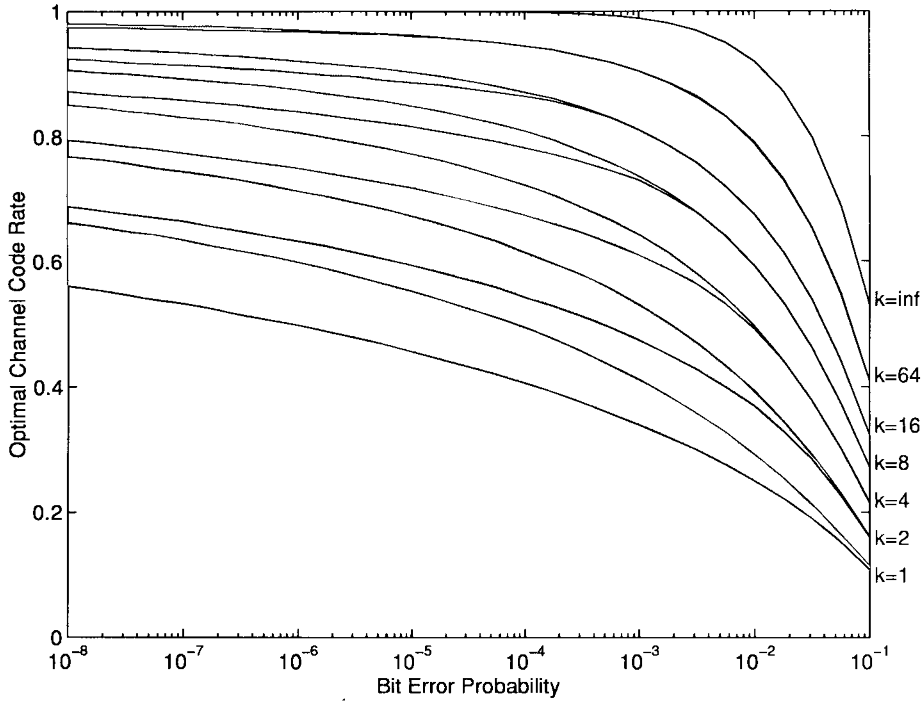

Fig. 2. Shaded regions between upper bound $r_{\rm sp}$ and lower bound $\max(r_{\rm ex}, r_{\rm ra})$ on channel code rate r that optimizes tradeoff between source and channel coding (inequality (29) with large channel transmission rate R, and $p = 2$) as a function of channel bit-error probability ϵ, for source vector dimensions $k = 1, 2, 4, 8, 16,$ and 64. The curve $k = \infty$ is the channel capacity.

Proof: See Appendix V.

Proof of Theorem 2: Because the upper bound in (24) does not depend on π, it follows that

$$\overline{D}_R(Q, \epsilon) \leq 2^{-pRr+O(1)} + 2^{-kRE_{\rm r}(r)+O(1)}.$$

Comparing this inequality with (14), we see that the r that minimizes the average distortion obeys

$$r_{\rm ra} \leq r \leq r_{\rm sp} \tag{28}$$

when R is sufficiently large. Lemmas 4 and 5 now complete the proof of Theorem 2. □

Remarks on Theorem 2: i) The corresponding distortion obeys

$$\overline{D}_R(Q, \epsilon) = 2^{-pRr+O(1)} = 2^{-pR[C-\alpha/\sqrt{k}+O(1/k)+o(1)]}$$

independently of the probability density function of the source vector.

ii) The requirement $k > p(\beta - \beta \log \beta - \gamma)/(\gamma - \beta \log \beta)$ becomes restrictive as $\epsilon \to 0$, since, for small ϵ, this requirement is approximately $k > p/(\sqrt{\epsilon} \log 1/\sqrt{\epsilon})$, the right-hand side of which can be quite large. For example, when $\epsilon = 10^{-5}$ and $p = 2$, then $p/(\sqrt{\epsilon} \log 1/\sqrt{\epsilon}) \approx 76$.

iii) Theorem 2 is consistent with the fact that the optimal $r \to C$ as $k \to \infty$ for any R (via Shannon's rate distortion and channel coding theorems).

iv) In Theorem 2, ϵ is fixed and k is large, this case not being addressed by Theorem 1, where k is fixed and ϵ small. Therefore, Theorems 1 and 2 complement each other on their applicable range of ϵ.

v) The proof of Theorem 1 uses inequality (20) which is valid for large R and for *all* ϵ and k. The proof of Theorem 2 uses inequality (28), which is also valid for large R and for all ϵ and k. Hence, for all ϵ and k, the channel code rate r that minimizes the average distortion satisfies

$$\max(r_{\rm ex}, r_{\rm ra}) \leq r \leq r_{\rm sp} \tag{29}$$

when R is sufficiently large. This inequality is used in some of the illustrations given in the next section.

V. Illustrations of Optimal Channel Code Rate

In this section, we complement the analytic small ϵ and large k bounds on the optimal channel code rate given in Theorems 1 and 2 with plots of the exact bounds, without the approximations in ϵ and k. All of the plots assume the standard squared-error distortion $p = 2$.

In Fig. 2, the upper and lower bounds on the optimal channel code rate r given in (29) are displayed as a function of bit-error probability ϵ for various values of source vector dimension k. The regions between the upper and lower bounds are shaded gray. To compute the regions, (5), (15), and (25) were solved numerically (using (3), (13), and (23)), and the $o(1)$ terms were ignored. Note that the optimal channel code rate is often substantially smaller than the channel capacity.

Another perspective of (29) is shown in Fig. 3, where the optimal channel code rate bounds are displayed as a function of k for various values of ϵ.

Fig. 4 shows (20) as a function of ϵ for $k = 3$. Also displayed are the analytic approximations presented in Theorem 1 and expanded more fully in Lemmas 1 and 3 (see (7) and (17)). As predicted, the analytic approximations become more accurate as ϵ decreases.

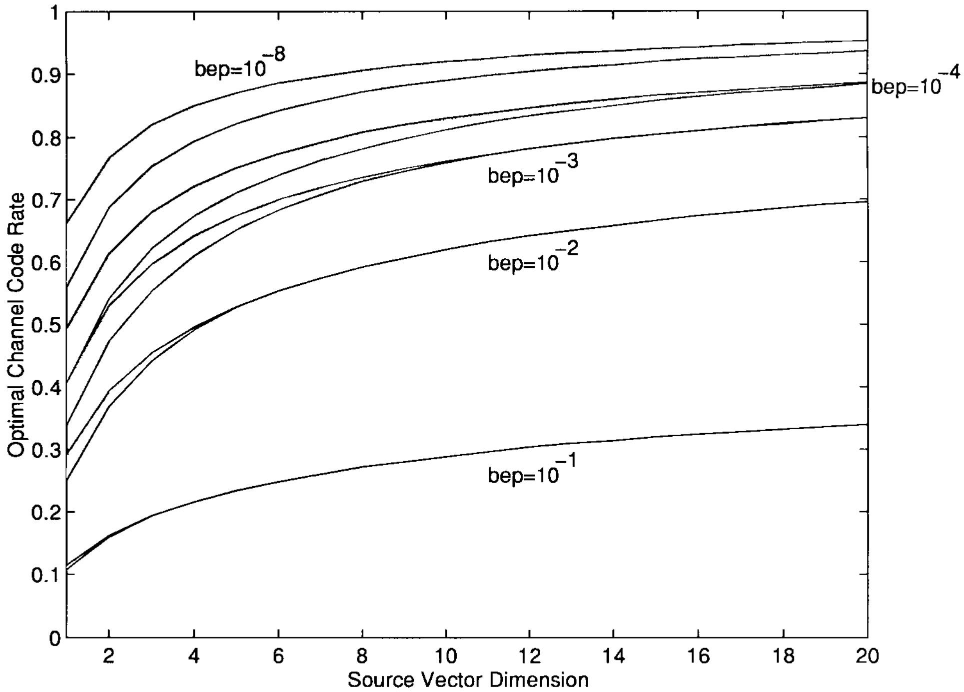

Fig. 3. Shaded regions between upper bound $r_{\rm sp}$ and lower bound $\max(r_{\rm ex},\, r_{\rm ra})$ on channel code rate r that optimizes tradeoff between source and channel coding (inequality (29) with large channel transmission rate R, and $p = 2$) as a function of source vector dimension k for channel bit-error probabilities $\epsilon = 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$, and 10^{-8}. For each ϵ, as $k \to \infty$, the upper and lower bounds eventually meet and then approach the channel capacity (via Theorem 2).

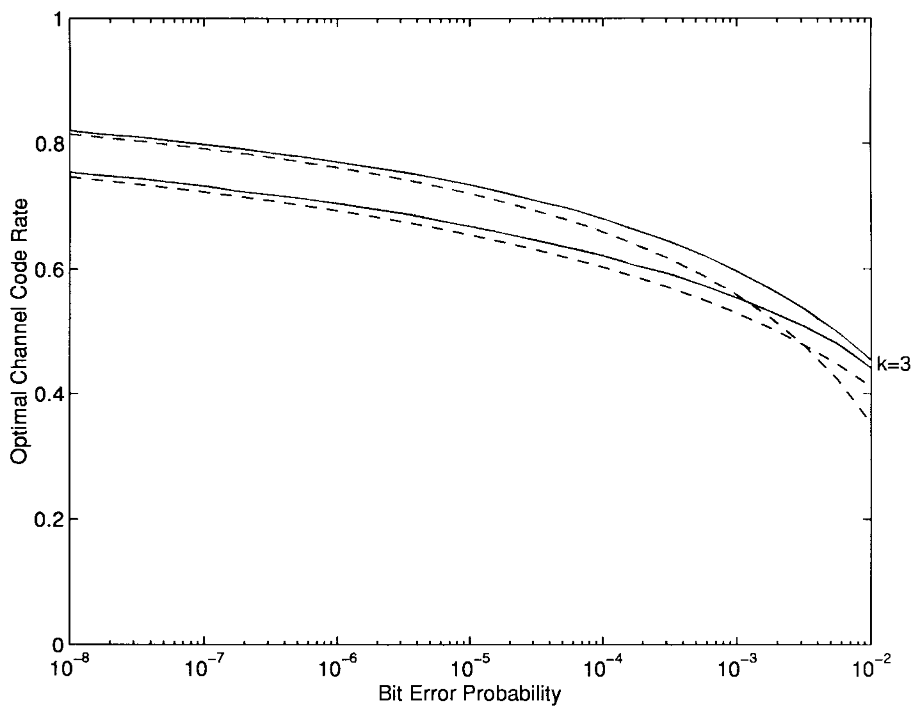

Fig. 4. Shaded region between upper bound $r_{\rm sp}$ and lower bound $r_{\rm ex}$ on channel code rate r that optimizes tradeoff between source and channel coding (inequality (20) with large channel transmission rate R, and $p = 2$) as a function of channel bit-error probability ϵ for $k = 3$. The dashed lines are the analytic approximations presented in Theorem 1 and expanded more fully in Lemmas 1 and 3.

From Figs. 2 and 3 it can be seen that, for large k, the upper and lower bounds meet $(r = r_{\rm sp} = \max(r_{\rm ex}, r_{\rm ra}))$. This fact is proven in Theorem 2, and in Fig. 5 we plot the optimal channel code rate (21) (omitting the $o(1)$ term) as a function of k for three different bit-error probabilities ϵ. The requirement that k exceed the threshold $2(\beta - \beta \log \beta - \gamma)/(\gamma - \beta \log \beta)$ is reflected in the starting values of k for each curve.

Fig. 6 plots the k threshold $2(\beta - \beta \log \beta - \gamma)/(\gamma - \beta \log \beta)$ given in (27), beyond which the large R upper and lower bounds on the optimal r meet, as a function of ϵ.

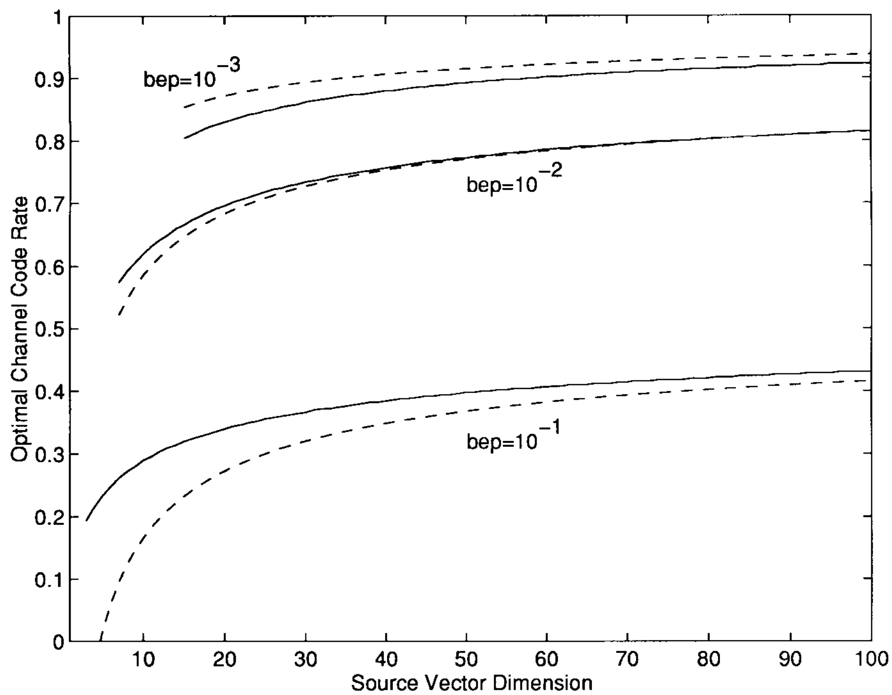

Fig. 5. Optimal channel code rate r (for large channel transmission rate R, and $p = 2$) as a function of the source vector dimension, k, for channel bit-error probabilities $\epsilon = 10^{-1}, 10^{-2}$, and 10^{-3}. Solid lines represent solutions to (25), and dashed lines represent analytic approximations (21) given in Theorem 2.

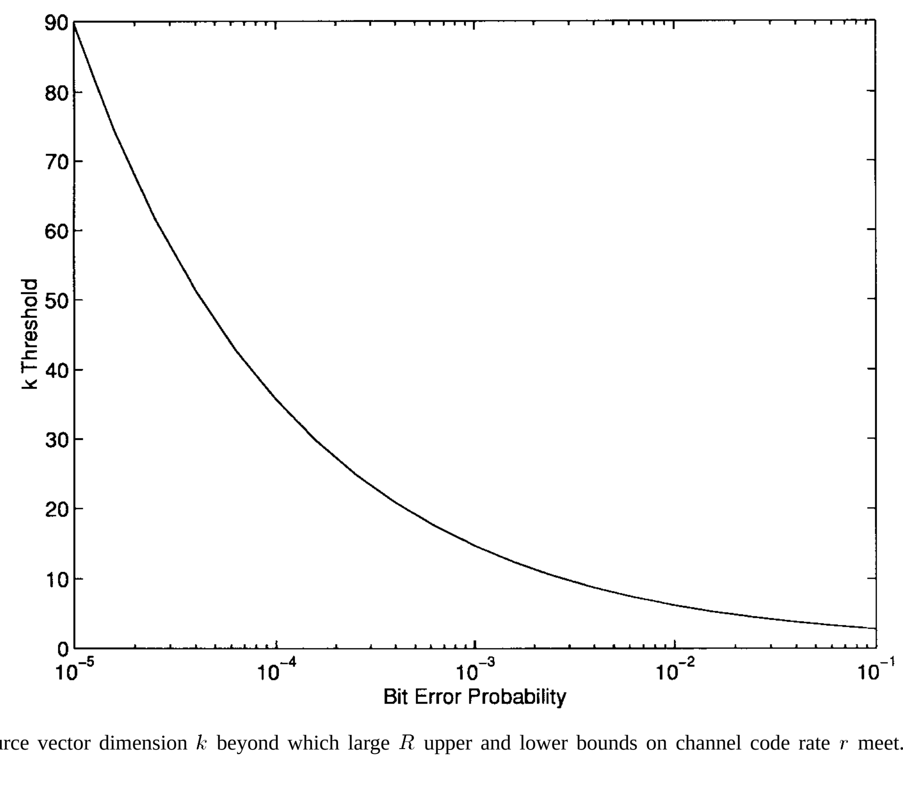

Fig. 6. Threshold on source vector dimension k beyond which large R upper and lower bounds on channel code rate r meet.

VI. Conclusion

We have derived the tradeoff between lossy source coding and block channel coding for a binary-symmetric channel. Tight bounds on the optimal channel code rate that minimized average distortion were provided. Analytical expressions were obtained for arbitrary source vector dimension and small channel error probability, and arbitrary channel error probability and large source dimension.

These bounds were derived by balancing the source and channel coding error exponents, and indicate the best performance that can be expected from a cascaded source and channel coder. To realize this performance, channel codes that have error exponents at least as good as the expurgated and random coding exponents are needed.

This paper has assumed that the m-bit source indices were individually channel coded and transmitted to the receiver. One may consider grouping a certain number of source indices together to increase the overall channel codeword blocklength. Shannon's coding theorem would then assure a decrease in the channel codeword error probability. It would then be straightforward to modify the results of Sections III and IV to decide the best channel code rate to use as a function of the number of grouped indices.

APPENDIX I
PROOF OF LEMMA 1

For any $r, \delta \in (0,1)$, the function

$$\rho[1 - r - \log(1 + \delta^{1/\rho})] \tag{30}$$

is concave for $\rho \geq 1$ since its second derivative with respect to ρ is $-\delta^{1/\rho}(\log^2 \delta)/[(1+\delta^{1/\rho})^2\rho^3 \ \log e] < 0$. Hence, this function can be maximized over $\rho \geq 1$ by finding the stationary point, assuming one exists. This proof proceeds by specifying ρ, and then deriving an r for which this ρ is a stationary point.

Define

$$\rho(\delta) \stackrel{\text{def}}{=} \frac{\log 1/\delta}{\log \log 1/\delta + c_\delta}.$$

Since $c_\delta \to \log k/p$ as $\delta \to 0$, it follows that $\rho(\delta) > 1$ for δ sufficiently small. Taking the derivative of (30) with respect to ρ, we conclude that any $r = r(\delta)$ for which $\rho(\delta)$ is a stationary point must satisfy

$$r(\delta) = 1 - \log(1 + \delta^{1/\rho(\delta)}) - \frac{1}{\rho(\delta)(1+\delta^{1/\rho(\delta)})}\delta^{1/\rho(\delta)} \ \log 1/\delta.$$

We have $\delta^{1/\rho(\delta)} = 2^{-c_\delta}/\log 1/\delta$, which goes to zero as δ goes to zero. Thus standard power series expansions yield

$$r(\delta) = 1 - \frac{2^{-c_\delta} \ \log e}{\log 1/\delta} + g(\delta) - \frac{2^{-c_\delta}(\log \log 1/\delta + c_\delta)}{\log 1/\delta} \cdot \left[1 - \frac{2^{-c_\delta}}{\log 1/\delta} + O\left(\frac{1}{\log^2 1/\delta}\right)\right] \tag{31}$$

where $g(\delta) = O(1/\log^2 1/\delta)$. Because $\rho(\delta) > 1$ for δ sufficiently small, the constraint $\rho \geq 1$ in (3) is inactive. Therefore,

$$\begin{aligned}
E_{ex}(r(\delta)) &= \rho(\delta)\left[1 - r(\delta) - \log\left(1 + \frac{2^{-c_\delta}}{\log 1/\delta}\right)\right] \\
&= \frac{\log 1/\delta}{\log \log 1/\delta + c_\delta} \cdot \left[1 - r(\delta) - \frac{2^{-c_\delta} \ \log e}{\log 1/\delta} + g(\delta)\right] \\
&= 2^{-c_\delta}\left[1 - \frac{2^{-c_\delta}}{\log 1/\delta} + O\left(\frac{1}{\log^2 1/\delta}\right)\right] \\
&= 2^{-c_\delta} - \frac{2^{-2c_\delta}}{\log 1/\delta} + O\left(\frac{1}{\log^2 1/\delta}\right) \\
&= \frac{p}{k}\left[1 - 2^{-c_\delta}\left(\frac{\log \log 1/\delta + \log e + c_\delta}{\log 1/\delta}\right)\right] + O\left(\frac{1}{\log^2 1/\delta}\right) \\
&= (p/k)r(\delta) + O((\log \log 1/\delta)/\log^2 1/\delta)
\end{aligned}$$

where the third and the last equalities follow from (31), and the penultimate from (6).

We know that r_{ex} obeys (5), and hence

$$[E_{\text{ex}}(r(\delta)) - E_{\text{ex}}(r_{\text{ex}})] + (p/k)[r_{\text{ex}} - r(\delta)] = O((\log \log 1/\delta)/\log^2 1/\delta) + o(1)$$

where the $O((\log \log 1/\delta)/\log^2 1/\delta)$ term goes to zero as $\delta \to 0$, uniformly in R, and the $o(1)$ term goes to zero as $R \to \infty$. The two differences enclosed within brackets both have the same sign because $E_{\text{ex}}(r)$ is a decreasing function of r. Therefore,

$$r_{\text{ex}} = r(\delta) + O((\log \log 1/\delta)/\log^2 1/\delta) + o(1)$$

and the lemma is proven.

APPENDIX II
PROOF OF LEMMA 2

Let the minimum over $x \in K$ of $\Sigma_{i=1}^M \|x - y_i\|^p$ be achieved, for each M, at some $x_M \in K$. For any $\delta > 0$, let the closed ball of radius δ centered at x_M be denoted $B_\delta(x_M)$. Suppose that the lemma is false; that is,

$$\liminf_{M\to\infty} (1/M)\sum_{i=1}^{M} \|x_M - y_i\|^p = 0.$$

Then, for some subsequence M_j, only $o(M_j)$ quantizer codevectors fall outside the sets $K \cap B_\delta(x_{M_j})$, as $j \to \infty$. We proceed to obtain a contradiction.

For δ sufficiently small, because K has nonempty interior, there exists a closed ball of positive probability contained in K with some radius δ' and center x', and a sub-subsequence M_l, such that $B_{\delta'}(x') \cap B_\delta(x_{M_l})$ is empty for all l. Then the number of codevectors in $B_{\delta'}(x')$ is $o(M_l)$ as $l \to \infty$.

The number of codevectors in the closed ball $B' \stackrel{\text{def}}{=} B_{\delta'/2}(x')$ is also $o(M_l)$. Because the quantizer uses the nearest neighbor rule to partition K, and the number of codevectors in any subset of K must go to infinity as $l \to \infty$, we conclude that when l is sufficiently large, codevectors outside $B_{\delta'}(x')$ are not used to quantize the region B', and codevectors inside B' are not used to quantize the region outside $B_{\delta'}(x')$. We now modify the original quantizer Q to create a new quantizer Q' having $M_l + o(M_l)$ codevectors $\{y_i'\}$, by increasing the number of codevectors in B' by the same number of codevectors that Q contains in the shell $B'^c \cap B_{\delta'}(x')$. Then we arrange all the codevectors and cells in B' so as to minimize

$$D_{m_l'}(Q'|X \in B') \stackrel{\text{def}}{=} \frac{1}{P(B')}\sum_{i=1}^{M_l + o(M_l)} \int_{S_i' \cap B'} \|x - y_i'\|^p f(x) \, dx$$

subject to the constraint that Q' use only codevectors within B' to quantize the region B'. (In the above, $m_l' = \log[M_l + o(M_l)]$.) With $m_l = \log M_l$, we obtain

$$D_{m_l}(Q|X \in B') \geq D_{m_l'}(Q'|X \in B').$$

Equation (1) then implies that

$$\begin{aligned} D_{m_l}(Q) &= 2^{-pm_l/k+O(1)} \\ &= D_{m_l}(Q|X \in B')\ \Pr(X \in B') \\ &\quad + D_{m_l}(Q|X \notin B')\ \Pr(X \notin B') \\ &\geq D_{m_l'}(Q'|X \in B')\ \Pr(X \in B') \\ &\quad + D_{m_l}(Q|X \notin B')\ \Pr(X \notin B') \end{aligned}$$

as $l \to \infty$. But this contradicts Zador's formula [14], which, since there are $o(M_l)$ codevectors in B', implies that

$$D_{m_l'}(Q'|X \in B') = 2^{-(p/k)\log\{o(M_l)\}+O(1)} = 2^{-o(m_l)}.$$

Thus it must be that

$$\liminf_{M\to\infty} (1/M) \sum_{i=1}^{M} \|x_M - y_i\|^p > 0.$$

This proves the lemma.

Appendix III
Proof of Lemma 3

This proof is similar to the proof of Lemma 1, and is therefore slightly abbreviated. For any $r, \epsilon \in (0,1)$, the function

$$\rho(1-r) - (\rho+1)\ \log(\epsilon^{1/(1+\rho)} + (1-\epsilon)^{1/(1+\rho)}) \tag{32}$$

is concave for $\rho \geq 0$ since its second derivative is

$$-\frac{(1-\epsilon)^{1/(1+\rho)}\epsilon^{1/(1+\rho)}[\log \epsilon - \log(1-\epsilon)]^2}{[\epsilon^{1/(1+\rho)} + (1-\epsilon)^{1/(1+\rho)}]^2(1+\rho)^3 \log e}$$

which is negative. Therefore, any stationary point must be a maximum. Define

$$\rho(\epsilon) \stackrel{\text{def}}{=} \frac{\log 1/\epsilon}{\log\log 1/\epsilon + c_\epsilon} - 1.$$

Since $c_\epsilon \to \log k/p$ as $\epsilon \to 0$, it follows that $\rho(\epsilon) > 0$ for ϵ sufficiently small. Taking the derivative of (32) with respect to ρ, we conclude that any $r = r(\epsilon)$ for which $\rho(\epsilon)$ is a stationary point must satisfy

$$\begin{aligned} r(\epsilon) &= 1 - \log[\epsilon^{1/(1+\rho(\epsilon))} + (1-\epsilon)^{1/(1+\rho(\epsilon))}] \\ &\quad - \frac{1}{\epsilon^{1/(1+\rho(\epsilon))} + (1-\epsilon)^{1/(1+\rho(\epsilon))}} \\ &\quad \cdot \left[\frac{\epsilon^{1/(1+\rho(\epsilon))}}{1+\rho(\epsilon)}\ \log\frac{1}{\epsilon} + \frac{(1-\epsilon)^{1/(1+\rho(\epsilon))}}{1+\rho(\epsilon)}\ \log\frac{1}{1-\epsilon}\right]. \end{aligned} \tag{33}$$

We have

$$\epsilon^{1/(1+\rho(\epsilon))} = 2^{-c_\epsilon}/\log 1/\epsilon$$

which goes to zero as ϵ goes to zero. Furthermore,

$$(1-\epsilon)^{1/(1+\rho(\epsilon))} = 1 + O((\epsilon\ \log\log 1/\epsilon)/\log 1/\epsilon).$$

Thus standard power series expansions yield

$$\begin{aligned} r(\epsilon) &= 1 - \frac{2^{-c_\epsilon}\ \log e}{\log 1/\epsilon} + g(\epsilon) \\ &\quad - \left[1 - \frac{2^{-c_\epsilon}}{\log 1/\epsilon} + O\left(\frac{1}{\log^2 1/\epsilon}\right)\right] \\ &\quad \cdot \left[\frac{2^{-c_\epsilon}(\log\log 1/\epsilon + c_\epsilon)}{\log 1/\epsilon} + O\left(\frac{\epsilon \log\log 1/\epsilon}{\log 1/\epsilon}\right)\right] \end{aligned} \tag{34}$$

where

$$g(\epsilon) = O(1/\log^2 1/\epsilon).$$

Because the $O((\epsilon\ \log\log 1/\epsilon)/\log 1/\epsilon)$ remainder term is negligible in comparison with all other terms, it is dropped. From (13)

$$\begin{aligned} E_{\text{sp}}(r(\epsilon)) &= \rho(\epsilon)[1 - r(\epsilon)] - (\rho(\epsilon)+1)\log[\epsilon^{1/(1+\rho(\epsilon))} \\ &\quad + (1-\epsilon)^{1/(1+\rho(\epsilon))}] \\ &= \left(\frac{\log 1/\epsilon}{\log\log 1/\epsilon + c_\epsilon} - 1\right) \\ &\quad \cdot \frac{2^{-c_\epsilon}(\log\log 1/\epsilon + c_\epsilon)}{\log 1/\epsilon} \\ &\quad \cdot \left[1 - \frac{2^{-c_\epsilon}}{\log 1/\epsilon} + O\left(\frac{1}{\log^2 1/\epsilon}\right)\right] \\ &\quad - \frac{2^{-c_\epsilon}\log e}{\log 1/\epsilon} + g(\epsilon) \\ &= 2^{-c_\epsilon}\left[1 - \frac{\log\log 1/\epsilon + \log e + c_\epsilon + 2^{-c_\epsilon}}{\log 1/\epsilon}\right] \\ &\quad + O\left(\frac{\log\log 1/\epsilon}{\log^2 1/\epsilon}\right) \\ &= \frac{p}{k}\left[1 - 2^{-c_\epsilon}\frac{\log\log 1/\epsilon + \log e + c_\epsilon}{\log 1/\epsilon}\right] \\ &\quad + O\left(\frac{\log\log 1/\epsilon}{\log^2 1/\epsilon}\right) \\ &= (p/k)r(\epsilon) + O((\log\log 1/\epsilon)/\log^2 1/\epsilon) \end{aligned}$$

where the penultimate equality follows from (16), and the last from (34).

The same argument used in the proof of Lemma 1 now shows that

$$r_{\text{sp}} = r(\epsilon) + O((\log\log 1/\epsilon)/\log^2 1/\epsilon) + o(1)$$

where the $o(1)$ term goes to zero as $R \to \infty$.

Appendix IV
Proof of Lemma 4

In Appendix III it is argued that

$$\rho(1-r) - (\rho+1)\ \log[\epsilon^{1/(1+\rho)} + (1-\epsilon)^{1/(1+\rho)}]$$

is a concave function of ρ, and therefore any stationary point must be a maximum. Define

$$\rho(k) = \frac{1}{\sqrt{k}}\left[\frac{2pC\ \log e}{\epsilon\ \log^2\epsilon + (1-\epsilon)\ \log^2(1-\epsilon) - H^2(\epsilon)}\right]^{1/2}$$

where

$$H(\epsilon) = -\epsilon\ \log\epsilon - (1-\epsilon)\ \log(1-\epsilon)$$

is the binary entropy. Then, it follows from (33) that any $r = r(k)$ for which $\rho(k)$ is a stationary point must satisfy

$$r(k) = 1 - \log\left[\epsilon^{1/(1+\rho(k))} + (1-\epsilon)^{1/(1+\rho(k))}\right] - \frac{1}{\epsilon^{1/(1+\rho(k))} + (1-\epsilon)^{1/(1+\rho(k))}} \cdot \left[\frac{\epsilon^{1/(1+\rho(k))}}{1+\rho(k)} \log\frac{1}{\epsilon} + \frac{(1-\epsilon)^{1/(1+\rho(k))}}{1+\rho(k)} \log\frac{1}{1-\epsilon}\right]. \tag{35}$$

We are interested in an asymptotic expansion of the right-hand side of (35) for small $\rho(k)$, or large k. First observe that

$$\epsilon^{1/(1+\rho(k))} = \epsilon[1 - \rho(k)\ \ln\ \epsilon + \rho^2(k)(\ln\ \epsilon + (1/2)\ \ln^2\ \epsilon)] + O(\rho^3(k))$$

as $k \to \infty$, and there is a similar expansion for $(1-\epsilon)^{1/(1+\rho(k))}$. Therefore,

$$\epsilon^{1/(1+\rho(k))} + (1-\epsilon)^{1/(1+\rho(k))} = 1 + \rho(k)H_e(\epsilon) + \rho^2(k)[-H_e(\epsilon) + (1/2)\epsilon\ \ln^2\ \epsilon + (1/2)(1-\epsilon)\ \ln^2\ (1-\epsilon)] + O(\rho^3(k))$$

where $H_e(\epsilon) = H(\epsilon)/\log e$, and, furthermore,

$$\log[\epsilon^{1/(1+\rho(k))} + (1-\epsilon)^{1/(1+\rho(k))}] = \rho(k)H(\epsilon) - \rho^2(k)\Big[H(\epsilon) + (1/(2\ \log\ e))[H^2(\epsilon) - \epsilon\ \log^2\epsilon - (1-\epsilon)\ \log^2(1-\epsilon)]\Big] + O(\rho^3(k)). \tag{36}$$

Equation (35) therefore becomes

$$\begin{aligned} r(k) &= 1 - \rho(k)H(\epsilon) + O(\rho^2(k)) + \frac{1}{1+\rho(k)H_e(\epsilon) + O(\rho^2(k))} \cdot \left[\frac{\epsilon[1-\rho(k)\ \ln\ \epsilon + O(\rho^2(k))]}{1+\rho(k)} \log\ \epsilon + \frac{(1-\epsilon)[1-\rho(k)\ \ln\ (1-\epsilon) + O(\rho^2(k))]}{1+\rho(k)} \cdot \log(1-\epsilon)\right] \\ &= 1 - \rho(k)H(\epsilon) + [1-\rho(k)H_e(\epsilon) + O(\rho^2(k))]\Big[[\epsilon\ \log\ \epsilon][1-\rho(k)(1+\ln\ \epsilon)] + [(1-\epsilon)\ \log(1-\epsilon)][1-\rho(k)(1+\ln(1-\epsilon))] + O(\rho^2(k))\Big] + O(\rho^2(k)) \\ &= 1 - H(\epsilon) - \rho(k)H(\epsilon) + \rho(k)\Big[H(\epsilon) - (1/\log e)[\epsilon\ \log^2\epsilon + (1-\epsilon)\ \log^2(1-\epsilon) - H^2(\epsilon)]\Big] + O(\rho^2(k)) \\ &= 1 - H(\epsilon) - (\rho(k)/\log e)[\epsilon\ \log^2\epsilon + (1-\epsilon)\ \log^2(1-\epsilon) - H^2(\epsilon)] + O(\rho^2(k)) \qquad (37) \\ &= C - (1/\sqrt{k})[(2pC/\log\ e)(\epsilon\ \log^2\epsilon + (1-\epsilon) \cdot \log^2(1-\epsilon) - H^2(\epsilon))]^{1/2} + O(1/k) \qquad (38) \end{aligned}$$

as $k \to \infty$.

For k sufficiently large, $\rho(k) < 1$. Hence, from (23) and (36), we obtain

$$\begin{aligned} E_r(r(k)) &= \rho(k)[1-r(k)] - (\rho(k)+1)[\rho(k)H(\epsilon) - \rho^2(k)\Big[H(\epsilon) - (1/(2\ \log\ e))(\epsilon\ \log^2\epsilon + (1-\epsilon)\ \log^2(1-\epsilon) - H^2(\epsilon))\Big] + O(\rho^3(k)) \\ &= \rho(k)[1-r(k)] - \rho(k)H(\epsilon) - (\rho^2(k)/(2\ \log\ e))[\epsilon\ \log^2\epsilon + (1-\epsilon)\log^2(1-\epsilon) - H^2(\epsilon)] + O(\rho^3(k)) \\ &= (\rho^2(k)/(2\ \log\ e))[\epsilon\ \log^2\epsilon + (1-\epsilon)\log^2(1-\epsilon) - H^2(\epsilon)] + O(\rho^3(k)) \\ &= (p/k)C + O(1/k^{3/2}) \\ &= (p/k)r(k) + O(1/k^{3/2}) \end{aligned}$$

where the third equality follows from (37), and the last from (38).

An argument similar to the one used in the proof of Lemma 1 now shows that

$$r_{\text{ra}} = r(k) + O(1/k) + O(1/R).$$

Appendix V
Proof of Lemma 5

Let

$$\beta = \sqrt{\epsilon} + \sqrt{1-\epsilon}$$

and

$$\gamma = -\sqrt{\epsilon}\ \log\ \sqrt{\epsilon} - \sqrt{1-\epsilon}\ \log\ \sqrt{1-\epsilon}.$$

We first show that $E_{\text{sp}}(r) = E_r(r)$ when $r \geq 1 - \log\ \beta - \gamma/\beta$. Observe that $E_{\text{sp}}(r) = E_r(r)$ as long as the maximizing ρ's in (13) and (23) are the same. It is easy to show that, as r increases, the maximizing ρ decreases; we therefore seek the smallest r, sometimes known as the critical rate [4], for which the maximizing ρ equals one. From (35) we see that the maximizing ρ equals one when

$$r = 1 - \log\ \beta - \gamma/\beta.$$

As k increases, the r_k satisfying $E_{\text{sp}}(r_k) = (p/k)r_k$ increases. Hence, for some critical value k', we have $r_k \geq 1 - \log\ \beta - \gamma/\beta$ when $k > k'$. From (13)

$$E_{\text{sp}}(1 - \log\ \beta - \gamma/\beta) = 1 - (1 - \log\ \beta - \gamma/\beta) - 2\ \log\ \beta = -\log\ \beta + \gamma/\beta.$$

Solving

$$E_{\text{sp}}(1 - \log\ \beta - \gamma/\beta) = (p/k')(1 - \log\ \beta - \gamma/\beta)$$

for k', we obtain

$$k' = p(\beta - \beta\ \log\ \beta - \gamma)/(\gamma - \beta\ \log\ \beta).$$

Hence, for $k > k', E_{\text{sp}}(r_k) = E_r(r_k) = (p/k)r_k$.

Since $E_{\text{sp}}(r_{\text{sp}}) = (p/k)r_{\text{sp}} + o(1)$ as $R \to \infty$, it follows that

$$[E_{\text{sp}}(r_k) - E_{\text{sp}}(r_{\text{sp}})] + (p/k)[r_{\text{sp}} - r_k] = o(1).$$

The two differences enclosed within brackets both have the same sign because $E_{\rm sp}(r)$ is a decreasing function of r. Therefore, $r_{\rm sp} = r_k + o(1)$. Because $k > k'$ and

$$E_{\rm r}(r_{\rm ra}) = (p/k)r_{\rm ra} + O(1/R)$$

as $R \to \infty$, it follows that both r_k and $r_{\rm ra}$ are at least $1-\log \beta - \gamma/\beta$ for R large enough. Therefore, $r_{\rm ra} = r_k + o(1)$, and, hence, $r_{\rm sp} = r_{\rm ra} + o(1)$.

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