Quantizers With Uniform Decoders and Channel-Optimized Encoders

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***Abstract*—Scalar quantizers with uniform decoders and channel-optimized encoders are studied for a uniform source on $[0, 1]$ and binary symmetric channels. Two families of affine index assignments are considered: the complemented natural code (CNC), introduced here, and the natural binary code (NBC). It is shown that the NBC never induces empty cells in the quantizer encoder, whereas the CNC can. Nevertheless, we show that the asymptotic distributions of quantizer encoder cells for the NBC and the CNC are equal and are uniform over a proper subset of the source's support region. Empty cells act as a form of implicit channel coding. An effective channel code rate associated with a quantizer designed for a noisy channel is defined and computed for the codes studied. By explicitly showing that the mean-squared error (MSE) of the CNC can be strictly smaller than that of the NBC, we also demonstrate that the NBC is suboptimal for a large range of transmission rates and bit error probabilities. This contrasts with the known optimality of the NBC when either both the encoder and decoder are not channel optimized, or when only the decoder is channel optimized.**

***Index Terms*—Data compression, index assignment, quantization, source channel coding.**

I. Introduction

One approach to improving the performance of a quantizer that transmits across a noisy channel is to design the quantizer's encoder and/or decoder to specifically take into account the statistics of the transmission channel. Necessary optimality conditions for such channel-optimized encoders and decoders were given, for example, in [2], [11], [12]. Alternatively, an explicit error control code can be cascaded with the quantizer, at the expense of added transmission rate. Additionally, the choice of index assignment in mapping source codewords to channel codewords can increase the performance of a quantization system with a noisy channel. Examples of index assignments include the natural binary code (NBC), the folded binary code, and the Gray code.

Ideally, one seeks a complete theoretical understanding of the structure and performance of a quantizer that transmits across a noisy channel, and whose encoder and decoder are channel optimized. Unfortunately, other than the optimality conditions given in [11], virtually no other analytical results are known regarding such quantizers. Quantizer design and performance with index assignments for general encoders and decoders (i.e., not necessarily channel optimized) was considered in [7], [16]. Experimentally, it was observed in [4], [5] that quantizers with both channel-optimized encoders and decoders can have empty cells, which serve as a form of implicit channel coding. Some theoretical results are known, however, when the quantizer has no channel optimization, or when only the quantizer decoder is channel optimized.

For uniform scalar quantizers with neither channel-optimized encoders nor decoders and with no explicit error control coding, formulas for the mean-squared error with uniform sources were given in [8], [9] for the NBC, the Gray code, and for randomly chosen index assignments on a binary symmetric channel. They also asserted (without a published proof) the optimality of the NBC for the binary symmetric channel. Crimmins *et al.* [1] proved the optimality of the NBC as asserted in [8], [9], and McLaughlin, Neuhoff, and Ashley [15] generalized this result to uniform vector quantizers. Various other analytical results on index assignments without channel-optimized encoders or decoders have been given in [10], [13], [14].

Quantizers with uniform encoders and channel-optimized decoders on binary symmetric channels were studied in [3]. For such quantizers, exact descriptions of the decoders were computed, and the asymptotic distributions of codepoints were determined for various index assignments. Distortions were calculated and compared to those of quantizers without channel optimization. The proof in [15] of the optimality of the NBC for quantizers with no channel optimization was extended in [3] to show that the NBC is also optimal for quantizers with uniform encoders and channel-optimized decoders.

In the present paper, we examine quantizers with uniform decoders and channel-optimized encoders operating over binary symmetric channels. In particular, we investigate a previously studied index assignment, namely, the NBC. In addition, we introduce a new affine index assignment which we call the complemented natural code (CNC) and which turns out to have a number of interesting properties. We specifically analyze the entropy of the encoder output in such quantizers, the high-resolution distribution of their encoding cells (i.e., the cell density function), and the mean-squared errors (MSEs) the quantizers achieve. We calculate a quantity we call the "effective channel code rate," which describes implicit channel coding, viewed in terms of the entropy of the encoder output. We also show that the NBC optimality results of [1], [3], [15] do not extend to quantizers with uniform decoders and channel-optimized encoders. In fact, the CNC is shown to perform better than the NBC.

Our main results for quantizers with uniform decoders and channel-optimized encoders are the following. For a uniform

Manuscript received April 14, 2004; revised November 1, 2005. This work was supported in part by Ericsson and the National Science Foundation.

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Communicated by M. Effros, Associate Editor for Source Coding.

Digital Object Identifier 10.1109/TIT.2005.862087

source on $[0,1]$ and a binary symmetric channel with bit error probability $\epsilon \in [0,1/2)$, we compute the effective channel code rates and cell densities for the NBC and CNC. It is shown that the NBC index assignment never induces empty cells (Corollary III.2), and the cell density function generated by the NBC is the same as the point density function for quantizers with uniform encoders and channel-optimized decoders with the NBC (Theorem III.4). In contrast, it is shown that the CNC can induce many empty cells (Corollary IV.3). However, the cell density functions generated by the CNC and the NBC are both uniform over the same interval (Theorem IV.5). We also show that the cell density function generated by the CNC is the same as the point density function for quantizers with uniform encoders and channel-optimized decoders with both the CNC and the NBC (Theorem IV.6). Then we extend a result in [8] by computing the MSE resulting from the NBC (Theorem V.3). As a comparison, we state the previously known MSE formula for channel unoptimized encoders with the NBC (Theorem V.2). Finally, we show that the NBC is suboptimal for quantizers with uniform decoders and channel-optimized encoders for many bit error probabilities (Theorem V.6).

We restrict attention in this paper to a uniform source on $[0,1]$. However, it will be apparent that the results can be generalized to any bounded interval on the real line.

The paper is organized as follows. Section II gives definitions and notation. Sections III and IV, respectively, give results for the NBC and CNC. Section V gives distortion analysis. Appendices I–IV contain the proofs of all lemmas and of selected theorems as well as various lemma statements.

II. Preliminaries

For any set S of reals, let $\overline{S}$ denote its closure. If S is an interval, let $l(S)$ denote its length. Let $\emptyset$ denote the empty set. Throughout this paper, "log" will mean logarithm base two.

A *rate n quantizer* on $[0,1]$ is a mapping

$$\mathcal{Q} : [0,1] \longrightarrow \{y_n(0), y_n(1), \ldots, y_n(2^n-1)\}.$$

Throughout this paper, all quantizers will be on the interval $[0,1]$ and we will assume $n \geq 2$. The real-valued quantities $y_n(i)$ are called *codepoints* and the set $\{y_n(0), \ldots, y_n(2^n-1)\}$ is called a *codebook*. For a noiseless channel, the quantizer $\mathcal{Q}$ is the composition of a *quantizer encoder* and a *quantizer decoder*. These are, respectively, mappings

$$\mathcal{Q}_e : [0,1] \longrightarrow \{0, 1, \ldots, 2^n-1\}$$
$$\mathcal{Q}_d : \{0, \ldots, 2^n-1\} \longrightarrow \{y_n(0), \ldots, y_n(2^n-1)\}$$

such that $\mathcal{Q}_d(i) = y_n(i)$ for all i. On a discrete, memoryless, noisy channel a quantizer is a composition of the quantizer encoder, the channel, and the quantizer decoder.

Without channel noise it is known that for an optimal quantizer, the encoder $\mathcal{Q}_e$ is a surjective mapping. However, in the presence of channel noise, it is possible that in an optimal quantizer the range of $\mathcal{Q}_e$ may contain fewer than 2^n points.

For each i, the ith encoding *cell* is the set

$$R_n(i) = \mathcal{Q}_e^{-1}(i).$$

If $R_n(i) = \emptyset$ we say $R_n(i)$ is an *empty cell*.

A quantizer with empty cells can be thought of as implicitly using channel coding to protect against channel noise. For example, if one half of the cells of a quantizer were empty, and the other half all had equal sizes, this could be thought of as effectively using one bit of error protection. More generally, the cascade of a rate k quantizer having 2^k equal size cells with an (n,k) block channel code can equivalently be viewed as a rate n quantizer with 2^n cells, 2^k of which are nonempty. That is, for any input lying in one of the 2^k nonempty cells, the k-bit index produced by the original quantizer encoder is expanded to n bits, which is then used for transmission. A quantizer can also introduce redundancy by making some encoding cells smaller than others. This reduces the entropy of the encoder output while maintaining the same transmission rate. To quantify the amount of natural error protection embedded in quantizers designed for noisy channels, we define the *effective channel code rate* of a quantizer as

$$r_c = \frac{H(\mathcal{Q}_e(X))}{n}$$

where X is a real-valued source random variable and H denotes the Shannon entropy. Then

$$0 \leq r_c \leq \frac{\log\left|\{i : R_n(i) \neq \emptyset\}\right|}{n} \leq 1.$$

In particular, the effective channel code rate of a rate k quantizer, having no empty cells, cascaded with an (n,k) block channel code (viewed as a rate n quantizer) is at most k/n, i.e., the rate of the channel code. For such a cascaded system, if r_b denotes the rate of the block channel code and if cell sizes are equal, then

$$r_c = r_b.$$

In this paper, we compute the effective channel code rates of certain quantizers that cannot be decomposed as cascades of (lower transmission rate) quantizers with block channel codes.

A quantizer encoder is said to be *uniform* if for each i, the ith cell satisfies

$$R_n(i) \supset (i2^{-n}, (i+1)2^{-n}).$$

We say the quantizer decoder is uniform, if for each i, the ith codepoint satisfies

$$y_n(i) = \left(i + \frac{1}{2}\right) 2^{-n}.$$

The *nearest neighbor* cells of a rate n quantizer are the sets

$$T_n(i) = \{x : |y_n(i) - x| < |y_n(j) - x|, \quad \forall j \neq i\}$$

for $0 \leq i \leq 2^n - 1$. A quantizer's encoder is said to satisfy the *nearest neighbor condition* if for each i

$$T_n(i) \subset R_n(i) \subset \overline{T}_n(i).$$

That is, its encoding cells are the nearest neighbor cells together with some boundary points (which can be assigned arbitrarily).

For given n, i, and real-valued source random variable X, the *centroid* of the ith cell of the quantizer $\mathcal{Q}$ is the conditional mean

$$c_n(i) = E[X|X \in R_n(i)].$$

The quantizer decoder is said to satisfy the *centroid condition* if the codepoints satisfy

$$y_n(i) = c_n(i)$$

for all i. A quantizer is *uniform* if both the encoder and decoder are uniform. It is known that if a quantizer minimizes the MSE for a given source and a noiseless channel, then it satisfies the nearest neighbor and centroid conditions [6]. In particular, if the source is uniform, then a uniform quantizer satisfies the nearest neighbor and centroid conditions.

For a rate n quantizer, an *index assignment* π_n is a permutation of the set $\{0, 1, \ldots, 2^n - 1\}$. Let S_n denote the set of all $2^n!$ such permutations. For a noisy channel, a random variable $X \in [0,1]$ is quantized by transmitting the index $I = \pi_n(\mathcal{Q}_e(X))$ across the channel, receiving index J from the channel, and then decoding the codepoint

$$y_n(\pi_n^{-1}(J)) = \mathcal{Q}_d(\pi_n^{-1}(J)).$$

The MSE is defined as

$$D = E\left[(X - \mathcal{Q}_d(\pi_n^{-1}(J)))^2\right]. \quad (1)$$

The random index J is a function of the source random variable X, the randomness in the channel, and the deterministic functions $\mathcal{Q}_e$ and π_n.

Assume a binary symmetric channel with bit error probability ϵ. Throughout this paper we use the notation

$$\delta = 1 - 2\epsilon$$
$$\omega = \frac{\epsilon}{1-\epsilon}.$$

Denote the probability that index j was received, given that index i was sent, by

$$p_n(j|i) = \epsilon^{H_n(i,j)}(1-\epsilon)^{n-H_n(i,j)}$$

for $0 \le \epsilon \le 1/2$, where $H_n(i,j)$ is the Hamming distance between n-bit binary words i and j. Let $q_n(i|j)$ denote the probability that index i was sent, given that index j was received.

For a given source X, channel $p_n(\cdot|\cdot)$, index assignment π_n, and quantizer encoder, the quantizer decoder is said to satisfy the *weighted centroid condition* if the codepoints satisfy

$$y_n(j) = \sum_{i=0}^{2^n-1} c_n(i) q_n(\pi_n(i)|\pi_n(j)).$$

For a given source X, channel $p_n(\cdot|\cdot)$, index assignment π_n, and quantizer decoder, the quantizer encoder is said to satisfy the *weighted nearest neighbor condition* if the encoding cells satisfy

$$W_i \subset R_n(i) \subset \overline{W}_i \quad (2)$$

where

$$W_i = \left\{ x : \sum_{j=0}^{2^n-1} (x - y_n(j))^2 p_n(\pi_n(j)|\pi_n(i)) < \sum_{j=0}^{2^n-1} (x - y_n(j))^2 p_n(\pi_n(j)|\pi_n(k)), \quad \forall k \neq i \right\}.$$

For a given quantizer encoder and index assignment, we say the quantizer has a *channel-optimized decoder* if it satisfies the weighted centroid condition. Similarly, for a given quantizer decoder and index assignment, we say the quantizer has a *channel-optimized encoder* if it satisfies the weighted nearest neighbor condition. It is known that a minimum MSE quantizer for a noisy channel must have both a channel-optimized encoder and decoder [11].

Lemma II.1: A quantizer with a uniform decoder and channel-optimized encoder satisfies, for all i

$$\overline{R}_n(i) = \{x \in [0,1] : \alpha_n(i,k)\, x \ge \beta_n(i,k), \quad \forall\, k \neq i\} \quad (3)$$

where

$$\alpha_n(i,k) = \sum_{j=0}^{2^n-1} j[p_n(\pi_n(j)|\pi_n(i)) - p_n(\pi_n(j)|\pi_n(k))] \quad (4)$$

$$\beta_n(i,k) = 2^{-n-1}\left(\alpha_n(i,k) + \sum_{j=0}^{2^n-1} j^2[p_n(\pi_n(j)|\pi_n(i)) - p_n(\pi_n(j)|\pi_n(k))]\right). \quad (5)$$

Lemma II.1 implies that each $R_n(i)$ is a (possibly empty) interval. Therefore, in this paper, when we describe quantizer encoding cells it suffices to describe their closures.

For any set A, denote the indicator function of A by

$$\chi_A(x) = \begin{cases} 1, & \text{for } x \in A \\ 0, & \text{for } x \notin A. \end{cases}$$

For a given quantizer encoder, let

$$\Lambda = \{i : R_n(i) \neq \emptyset\}.$$

These are the indices of nonempty cells.

For each n and each index assignment $\pi_n \in S_n$, define the function

$$\gamma_{\pi_n}^{(n)} : [0,1] \to [0,\infty)$$

by

$$\gamma_{\pi_n}^{(n)}(x) = \sum_{i \in \Lambda} \frac{1}{|\Lambda| \cdot l(R_n(i))} \chi_{R_n(i)}(x).$$

For a sequence $\pi_n \in S_n$ (for $n = 1, 2, \ldots$) of index assignments, if there exists a measurable function γ such that

$$\gamma(x) = \lim_{n\to\infty} \gamma_{\pi_n}^{(n)}(x)$$

for almost all $x \in [0,1]$ and

$$\int_0^1 \gamma(x)\, dx = 1$$

then we say γ is a *cell density function* with respect to $\{\pi_n\}$.

For each n and each index assignment $\pi_n \in S_n$, define the function

$$\lambda_{\pi_n}^{(n)} : [0,1] \to [0,\infty)$$

by

$$\lambda_{\pi_n}^{(n)}(x) = \sum_{i=0}^{2^n-1} \frac{1}{2^n \cdot l(T_n(i))} \chi_{T_n(i)}(x).$$

For a sequence $\pi_n \in S_n$ (for $n = 1, 2, \ldots$) of index assignments, if there exists a measurable function λ such that

$$\lambda(x) = \lim_{n\to\infty} \lambda_{\pi_n}^{(n)}(x)$$

for almost all $x \in [0,1]$ and

$$\int_0^1 \lambda(x)\, dx = 1$$

then we say λ is a *point density function* with respect to $\{\pi_n\}$.

The integrals $\int_a^b \gamma$ and $\int_a^b \lambda$ give the asymptotic fraction of encoding cells and decoder codepoints, respectively, that appear in the interval $[a,b]$ as $n \to \infty$.

Let a *decoder-optimized uniform quantizer* (DOUQ) denote a rate n quantizer with a uniform encoder on $[0,1]$ and a channel-optimized decoder, along with a uniform source on $[0,1]$, and a binary symmetric channel with bit error probability ϵ. When considering DOUQs, we impose the following monotonicity constraint on the quantizer encoder in order to be able to unambiguously refer to particular index assignments: For all $s, t \in [0,1]$, if $s < t$, then $\mathcal{Q}_e(s) \le \mathcal{Q}_e(t)$. In other words, the encoding cells are labeled from left to right.

Let an *encoder-optimized uniform quantizer* (EOUQ) denote a rate n quantizer with a uniform decoder and a channel-optimized encoder, along with a uniform source on $[0,1]$, and a binary symmetric channel with bit error probability ϵ. When considering EOUQs, we impose the following monotonicity constraint on the quantizer decoder in order to be able to unambiguously refer to particular index assignments: For any $y_n(i)$ and $y_n(j)$, if $y_n(i) < y_n(j)$, then $\mathcal{Q}_d^{-1}(y_n(i)) < \mathcal{Q}_d^{-1}(y_n(j))$. In other words, the codepoints are labeled in increasing order.

An alternative approach would be to view the quantizer encoder as the composition $\pi_n \cdot \mathcal{Q}_e$ and the quantizer decoder as the composition $\mathcal{Q}_d \cdot \pi_n^{-1}$, by relaxing the monotonicity assumptions made above. This would remove the role of index assignments from the study of quantizers for noisy channels. However, we retain these encoder and decoder decompositions, as a convenient way to isolate the effects of index assignments, given known quantizer encoders and decoders.

Let a *channel unoptimized uniform quantizer* denote a rate n uniform quantizer on $[0,1]$, along with a uniform source on $[0,1]$, and a binary symmetric channel with bit error probability ϵ.

III. NBC Index Assignment

For each n, the NBC is the index assignment defined by

$$\pi_n^{(\mathrm{NBC})}(i) = i, \qquad \text{for } 0 \le i \le 2^n - 1.$$

Theorem III.1: An EOUQ with the NBC index assignment has encoding cells given by

$$\overline{R}_n(i) = \begin{cases} \left[0, \epsilon + \delta 2^{-n}\right], & \text{for } i = 0 \\ \left[\epsilon + i\delta 2^{-n}, \epsilon + \delta(i+1)2^{-n}\right], & \text{for } 1 \le i \le 2^n - 2 \\ \left[1 - \epsilon - \delta 2^{-n}, 1\right], & \text{for } i = 2^n - 1. \end{cases}$$

Proof: The encoding cells satisfy (3) in Lemma II.1, with

$$\begin{aligned} \alpha_n(i,k) &= \sum_{j=0}^{2^n-1} j \left[p_n(\pi_n^{(NBC)}(j)|\pi_n^{(NBC)}(i)) - p_n(\pi_n^{(NBC)}(j)|\pi_n^{(NBC)}(k)) \right] \\ &= (i-k)\delta \qquad (6) \\ \beta_n(i,k) &= 2^{-n-1} \cdot \left(\alpha_n(i,k) + \sum_{j=0}^{2^n-1} j^2 \left[p_n(\pi_n^{(NBC)}(j)|\pi_n^{(NBC)}(i)) - p_n(\pi_n^{(NBC)}(j)|\pi_n^{(NBC)}(k)) \right] \right) \\ &= 2^{-n-1} \left[(i-k)\delta[1 + 2\epsilon(2^n - 1)] + (i^2 - k^2)\delta^2 \right] \qquad (7) \end{aligned}$$

where (6) follows from Lemma II.2 and (7) follows from (6) and Lemma II.3. Thus,

$$\frac{\beta_n(i,k)}{\alpha_n(i,k)} = \epsilon + \delta(i+k+1)2^{-n-1}, \qquad 0 \le i, k \le 2^n - 1. \quad (8)$$

From (6), we have that $\alpha_n(i,k) > 0$ if and only if $i > k$, and $\alpha_n(i,k) < 0$ if and only if $i < k$. Therefore, (3) can be rewritten as

$$\overline{R}_n(i) = \left\{ x \in [0,1] : x \ge \frac{\beta_n(i,k)}{\alpha_n(i,k)}, \quad \forall\ k < i \text{ and } x \le \frac{\beta_n(i,k)}{\alpha_n(i,k)}, \quad \forall\, k > i \right\}. \quad (9)$$

By (8), the quantity

$$\frac{\beta_n(i,k)}{\alpha_n(i,k)}$$

is increasing in both i and k. Hence, if $1 \le i \le 2^n - 1$, then (taking $k = i - 1$ in (9)) $x \in \overline{R}_n(i)$ if and only if

$$\begin{aligned} x &\ge \epsilon + \delta(i + (i-1) + 1)2^{-n-1} \\ &= \epsilon + i\delta 2^{-n}. \end{aligned}$$

Similarly, if $0 \le i \le 2^n - 2$, then (taking $k = i + 1$ in (9)) $x \in \overline{R}_n(i)$ if and only if

$$\begin{aligned} x &\le \epsilon + \delta(i + (i+1) + 1)2^{-n-1} \\ &= \epsilon + \delta(i+1)2^{-n}. \end{aligned} \quad (10)$$

□

A consequence of the preceding theorem is that the NBC produces no empty cells when the weighted nearest neighbor con-

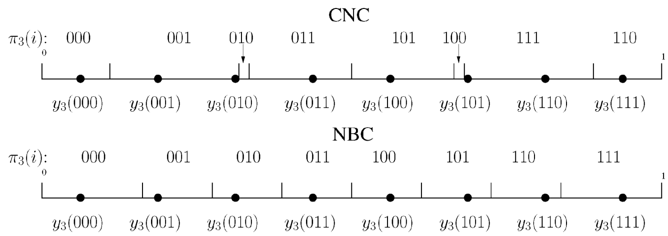

Fig. 1. Plot of the encoding cells of rate 3 EOUQs with the CNC and NBC index assignments and a bit error rate 0.05.

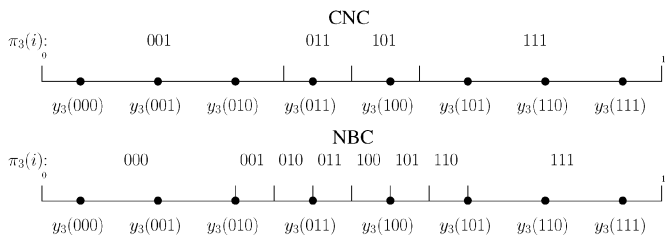

Fig. 2. Plot of the encoding cells of rate 3 EOUQs with the CNC and NBC index assignments and a bit error rate 0.25.

dition is used together with uniformly spaced codepoints. This fact is stated as the following result.

Corollary III.2: For all n and for all $\epsilon \in [0, 1/2)$, an EOUQ with the NBC index assignment has no empty cells.

Figs. 1 and 2 illustrate the encoding cells of a rate 3 EOUQ with the NBC index assignment for bit error rates 0.05 and 0.25, respectively. Fig. 3 plots the encoding cell boundaries of a rate 3 EOUQ with the NBC index assignment as a function of bit error rate.

Theorem III.3: An EOUQ with the NBC index assignment has an effective channel code rate given by

$$r_c = (1-2^{1-n})(1-2\epsilon)\left(1-\frac{\log(1-2\epsilon)}{n}\right) + \frac{2\epsilon+(1-2\epsilon)2^{1-n}}{n}\log\left(\frac{1}{\epsilon+(1-2\epsilon)2^{-n}}\right).$$

Proof: The definition of r_c implies

$$r_c = \frac{1}{n}\sum_{i\in\Lambda} l(R_n(i))\log\frac{1}{l(R_n(i))}.$$

From Theorem III.1

$$l(R_n(i)) = \begin{cases} \epsilon+\delta 2^{-n}, & \text{for } i=0 \\ \delta 2^{-n}, & \text{for } 1\le i\le 2^n-2 \\ \epsilon+\delta 2^{-n}, & \text{for } i=2^n-1. \end{cases}$$

Therefore,

$$\begin{aligned} r_c &= (2^n-2)\frac{\delta 2^{-n}}{n}\log\left(\frac{1}{\delta 2^{-n}}\right) + 2\frac{(\epsilon+\delta 2^{-n})}{n}\log\left(\frac{1}{\epsilon+\delta 2^{-n}}\right) \\ &= (1-2^{1-n})\delta\left(1-\frac{\log\delta}{n}\right) + \frac{2\epsilon+\delta 2^{1-n}}{n}\log\left(\frac{1}{\epsilon+\delta 2^{-n}}\right). \end{aligned}$$

□

As $n \to \infty$, the effective channel code rate given by Theorem III.3 converges to $1-2\epsilon$. Fig. 5 plots the quantity r_c from Theorem III.3 for rate $n=4$.

The following theorem shows that the cell density function for a sequence of EOUQs with the NBC is the same as the point density function found in [3] for a sequence of DOUQs with the NBC.

Theorem III.4: A sequence of EOUQs with the NBC index assignment has a cell density function given by

$$\gamma(x) = \begin{cases} \frac{1}{1-2\epsilon}, & \text{for } \epsilon < x < 1-\epsilon \\ 0, & \text{else.} \end{cases}$$

Proof: From Theorem III.1

$$l(R_n(i)) = \begin{cases} \epsilon+\delta 2^{-n}, & \text{for } i=0 \\ \delta 2^{-n}, & \text{for } 1\le i\le 2^n-2 \\ \epsilon+\delta 2^{-n}, & \text{for } i=2^n-1. \end{cases}$$

Therefore, since $|\Lambda| = 2^n$ by Corollary III.2

$$\begin{aligned} &\gamma^{(n)}_{\pi_n^{(\mathrm{NBC})}}(x) \\ &= \begin{cases} \frac{1}{\delta}, & \text{for } \epsilon+\delta 2^{-n} \le x < 1-\epsilon-\delta 2^{-n} \\ \frac{1}{\epsilon 2^n+\delta}, & \text{for } 0 \le x < \epsilon+\delta 2^{-n} \text{ or } \\ & \qquad 1-\epsilon-\delta 2^{-n} \le x \le 1 \end{cases} \\ &\longrightarrow \begin{cases} \frac{1}{\delta}, & \text{for } \epsilon < x < 1-\epsilon \\ 0, & \text{for } 0 \le x \le \epsilon \text{ or } 1-\epsilon \le x \le 1 \end{cases} \\ &\text{as } n \to \infty. \end{aligned}$$

□

IV. CNC Index Assignment

Let the CNC be the index assignment defined by

$$\pi_n^{(\mathrm{CNC})}(i) = \begin{cases} i, & \text{for } 0 \le i \le 2^{n-1}-1 \\ i+1, & \text{for } 2^{n-1} \le i \le 2^n-2 \\ & \text{and } i \text{ even} \\ i-1, & \text{for } 2^{n-1}+1 \le i \le 2^n-1 \\ & \text{and } i \text{ odd}. \end{cases}$$

Note that the CNC is a linear index assignment,[1] since

$$\pi_n^{(\mathrm{CNC})}(i) = iG_n$$

where i is an n-bit binary word, G_n is the $n \times n$ identity matrix with an additional 1 in the upper right-hand corner, and arithmetic is performed modulo 2 in the product iG_n. The CNC is closely related to the NBC. However, it induces very different encoding cell boundaries for EOUQs, as shown by Theorem IV.2.

Lemma IV.1: For each n, the polynomial

$$\phi_n(\epsilon) = -8\epsilon^3 + (4-2^{n+1})\epsilon^2 + (2+2^{n+1})\epsilon - 1$$

restricted to $\epsilon \in (0, 1/2)$ has a unique root ϵ_n^*. The polynomial is negative if and only if $\epsilon < \epsilon_n^*$. Furthermore, ϵ_n^* is monotonic decreasing and

$$\epsilon_n^* < (2^{n/2}+2)^{-1}.$$

The quantity ϵ_n^* defined in Lemma IV.1 will be frequently referenced throughout the remainder of the paper. ϵ_n^* plays an important role as a threshold value for the bit error probability of a binary symmetric channel, beyond which the encoding regions

[1] Affine index assignments were studied in [13]. The NBC and Gray code are linear, and the folded binary code is affine.

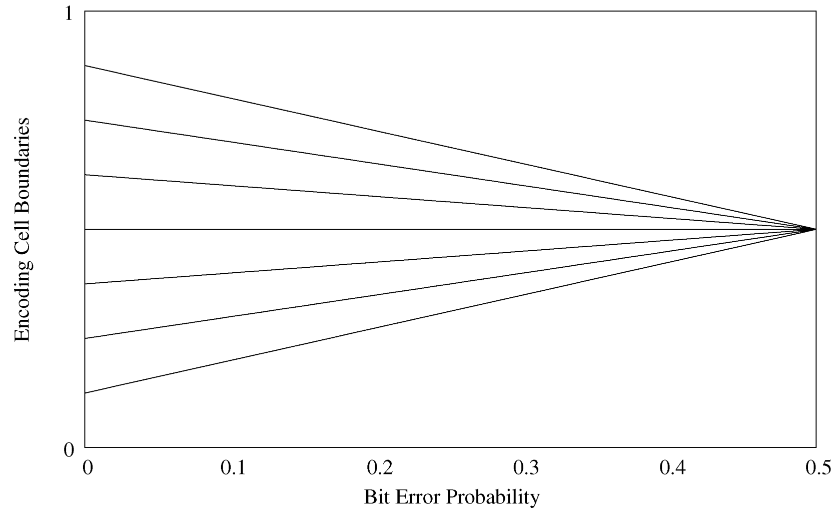

Fig. 3. Plot of the encoding cells boundaries of a rate 3 EOUQ with the NBC index assignment as a function of bit error rate.

and empty cells of an EOUQ with the CNC index assignment change in behavior. It can be shown, using the general solution to a cubic, that

$$\epsilon_n^* = \frac{2^n+4}{12}\cdot\left[\sqrt{3}\sin\left(\frac{\arctan(\tau/\sigma)+\pi}{3}\right) - \cos\left(\frac{\arctan(\tau/\sigma)+\pi}{3}\right) - 1\right] + \frac{1}{2}$$

where

$$\sigma = 2^{n-5} - \frac{1}{27}\left(2^{n-2}+1\right)^3$$
$$\tau = \sqrt{2^{n-4}(2^{n-6}-\sigma)}.$$

Theorem IV.2: The encoding cells of an EOUQ with the CNC index assignment are given as follows.

If $n=2$ and $\epsilon \in [0, 1/4)$, or if $n \geq 3$ and $\epsilon \in [0, \epsilon_n^*)$, then we have the equation at the bottom of the page. If $n=2$ and $\epsilon \in [1/4, 1/2)$, then

$$\overline{R}_n(i) = \begin{cases} \emptyset, & \text{for } i=0 \\ [0,1/2], & \text{for } i=1 \\ [1/2,1], & \text{for } i=2 \\ \emptyset, & \text{for } i=3. \end{cases}$$

If $n \geq 3$ and $\epsilon \in [\epsilon_n^*, 1/2)$, we have the equation at the bottom of the following page.

Corollary IV.3: For an EOUQ with the CNC index assignment, the number of nonempty cells is

$$|\Lambda| = \begin{cases} 2^n, & \text{for } \epsilon \in [0, \epsilon_n^*) \\ 2^{n-1}+2, & \text{for } \epsilon \in [\epsilon_n^*, 1/(2^{n/2}+2)) \\ 2^{n-1}, & \text{for } \epsilon \in [1/(2^{n/2}+2), 1/2). \end{cases}$$

If $\epsilon \in [\epsilon_n^*, 1/(2^{n/2}+2))$, then the indices of the empty cells are

$$\{i : 2 \leq i \leq 2^{n-1}-2,\ i \text{ even}\} \cup \{i : 2^{n-1}+1 \leq i \leq 2^n-3,\ i \text{ odd}\}.$$

If $\epsilon \in [1/(2^{n/2}+2), 1/2)$, then the indices of the empty cells are

$$\{i : 0 \leq i \leq 2^{n-1}-2,\ i \text{ even}\} \cup \{i : 2^{n-1}+1 \leq i \leq 2^n-1,\ i \text{ odd}\}.$$

Figs. 1 and 2 illustrate the encoding cells of a rate 3 EOUQ with the CNC index assignment for bit error rates 0.05 and 0.25, respectively. Fig. 4 plots the encoding cell boundaries of a rate 3 EOUQ with the CNC index assignment as a function of bit error rate.

$$\overline{R}_n(i) = \begin{cases} \left[0, \delta 2^{-n} - \frac{\epsilon^2}{\delta}\right], & \text{for } i=0 \\ \left[i\delta 2^{-n} - \frac{\epsilon^2}{\delta}, (i+1)\delta 2^{-n} + \frac{\epsilon(2+\epsilon)}{1+2\epsilon}\right], & \text{for } 1 \leq i \leq 2^{n-1}-3,\ i \text{ odd} \\ \left[i\delta 2^{-n} + \frac{\epsilon(2+\epsilon)}{1+2\epsilon}, (i+1)\delta 2^{-n} - \frac{\epsilon^2}{\delta}\right], & \text{for } 2 \leq i \leq 2^{n-1}-2,\ i \text{ even} \\ \left[(2^{-1}-2^{-n})\delta - \frac{\epsilon^2}{\delta}, 1/2\right], & \text{for } i = 2^{n-1}-1 \\ \left[1/2, (2^{-1}+2^{-n})\delta + \frac{\epsilon(2-3\epsilon)}{\delta}\right], & \text{for } i = 2^{n-1} \\ \left[i\delta 2^{-n} + \frac{\epsilon(2-3\epsilon)}{\delta}, (i+1)\delta 2^{-n} + \frac{3\epsilon^2}{1+2\epsilon}\right], & \text{for } 2^{n-1}+1 \leq i \leq 2^n-3,\ i \text{ odd} \\ \left[i\delta 2^{-n} + \frac{3\cdot\epsilon^2}{1+2\epsilon}, (i+1)\delta 2^{-n} + \frac{\epsilon(2-3\epsilon)}{\delta}\right], & \text{for } 2^{n-1}+2 \leq i \leq 2^n-2, i \text{ even} \\ \left[(1-2^{-n})\delta + \frac{\epsilon(2-3\epsilon)}{\delta}, 1\right], & \text{for } i = 2^n-1\,. \end{cases}$$

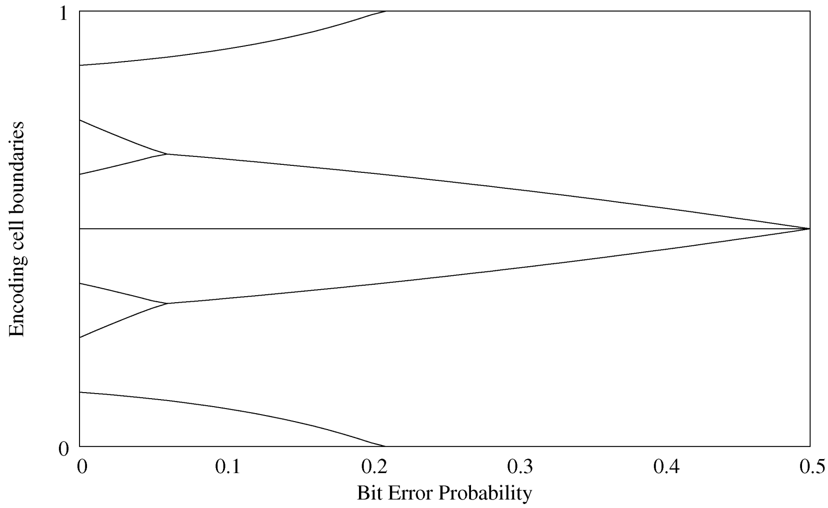

Fig. 4. Plot of the encoding cells boundaries of a rate 3 EOUQ with the CNC index assignment as a function of bit error rate.

Theorem IV.4: An EOUQ with the CNC index assignment has an effective channel code rate r_c given as follows. Let h be the binary entropy function and let

$$\begin{aligned} p_1 &= (1-2\epsilon)2^{-n} - \frac{\epsilon^2}{1-2\epsilon} \\ p_2 &= (1-2\epsilon)2^{-n} + \frac{\epsilon^2}{1-2\epsilon} + \epsilon \\ p_3 &= (1-2\epsilon)2^{-n} + \frac{2\epsilon(1-\epsilon)}{1-4\epsilon^2} \\ p_4 &= (1-2\epsilon)2^{-n} - \frac{2\epsilon(1-\epsilon)}{1-4\epsilon^2} \\ p_5 &= (1-2\epsilon)2^{-n} + 2^{-n-1}(1-2\epsilon)^2 + \frac{\epsilon^2}{1-2\epsilon} \\ p_6 &= (1-2\epsilon)2^{1-n} - 2^{-n-1}(1-2\epsilon)^2 \\ p_7 &= (1-2\epsilon)2^{1-n} + 2^{-n-1}(1-2\epsilon)^2 + \epsilon \\ p_8 &= \frac{1}{2}(1-2^{3-n})(1-2\epsilon)\left(n-1-\log(1-2\epsilon)\right). \end{aligned}$$

If $n=2$ and $\epsilon \in [0, 1/4)$, then

$$r_c = \frac{1}{2}\left(1 + h\left(\frac{1-2\epsilon}{2^{n-1}} - \frac{2\epsilon^2}{1-2\epsilon}\right)\right).$$

If $n=2$ and $\epsilon \in [1/4, 1/2)$, then $r_c = 1/2$.

If $n \geq 3$ and $\epsilon \in [0, \epsilon_n^*)$, then

$$r_c = -\frac{2}{n}\Big(p_1 \log p_1 + p_2 \log p_2 + (2^{n-2}-1)(p_3 \log p_3 + p_4 \log p_4)\Big).$$

If $n \geq 3$ and $\epsilon \in [\epsilon_n^*, 1/(2^{n/2}+2))$, then

$$r_c = -\frac{2}{n}\left(p_1 \log p_1 + p_5 \log p_5 + p_6 \log p_6 - p_8\right).$$

If $n \geq 3$ and $\epsilon \in [1/(2^{n/2}+2), 1/2)$, then

$$r_c = -\frac{2}{n}\left(p_7 \log p_7 - p_8 + p_6 \log p_6\right).$$

As $n \to \infty$, the effective channel code rate given by Theorem IV.4 converges to $1-2\epsilon$, for all $\epsilon \in [0, 1/2)$. Fig. 5 plots the quantity r_c from Theorem IV.4 for rate $n=4$.

Corollary IV.3 shows that given a bit error probability $\epsilon > 0$, for n sufficiently large, an EOUQ with the CNC has half the number of nonempty encoding cells as one with the NBC. The following theorem shows that despite this fact, for a sequence of EOUQs, the CNC and the NBC induce the same cell density function (via Theorem III.4).

Theorem IV.5: A sequence of EOUQs with the CNC index assignment has a cell density function given by

$$\gamma(x) = \begin{cases} \frac{1}{1-2\epsilon}, & \text{for } \epsilon < x < 1-\epsilon \\ 0, & \text{else.} \end{cases}$$

$$\overline{R}_n(i) = \begin{cases} \left[0, \delta 2^{-n} - \frac{\epsilon^2}{\delta}\right], & \text{for } i=0 \text{ and } \epsilon < 1/(2^{n/2}+2) \\ \left[\delta 2^{-n} - \frac{\epsilon^2}{\delta}, \left(4\delta+\delta^2\right)2^{-n-1}+\epsilon\right] \cap [0,1], & \text{for } i=1 \\ \left[\frac{2\delta(i-1)+\delta^2}{2^{n+1}}+\epsilon, \frac{2\delta(i+1)+\delta^2}{2^{n+1}}+\epsilon\right], & \text{for } 3 \leq i \leq 2^{n-1}-3,\ i \text{ odd} \\ \left[\left((2^n-4)\delta+\delta^2+2^{n+1}\epsilon\right)2^{-n-1}, 1/2\right], & \text{for } i = 2^{n-1}-1 \\ \left[1/2, \left((2^n+2)\delta+1-4\epsilon^2+2^{n+1}\epsilon\right)2^{-n-1}\right], & \text{for } i = 2^{n-1} \\ \left[\frac{2\delta(i-1)+1-4\epsilon^2}{2^{n+1}}+\epsilon, \frac{2\delta(i+1)+1-4\epsilon^2}{2^{n+1}}+\epsilon\right], & \text{for } 2^{n-1}+2 \leq i \leq 2^n-4,\ i \text{ even} \\ \left[\frac{(2^{n+1}-6)\delta+1-4\epsilon^2}{2^{n+1}}+\epsilon, (1-2^{-n})\delta+\frac{\epsilon(2-3\epsilon)}{\delta}\right] \cap [0,1], & \text{for } i = 2^n-2 \\ \left[\left((2^n-1)\delta+\frac{2^n\epsilon(2-3\epsilon)}{\delta}\right)2^{-n}, 1\right], & \text{for } i = 2^n-1 \text{ and } \epsilon < 1/(2^{n/2}+2) \\ \emptyset, & \text{else.} \end{cases}$$

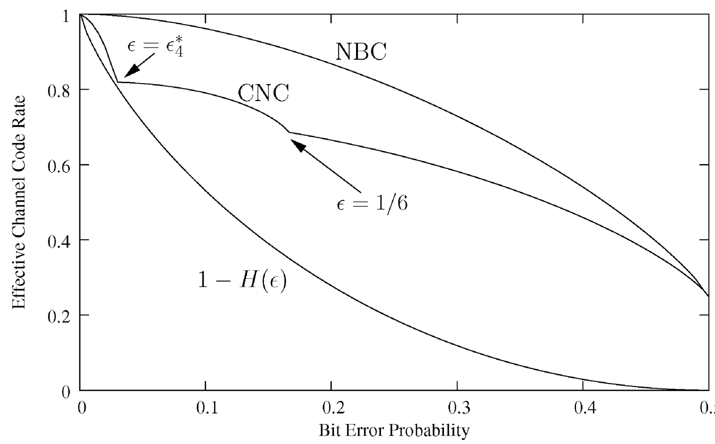

Fig. 5. Plot of the effective channel code rate r_c of EOUQs with the NBC and the CNC index assignments for rate $n = 4$. The horizontal axis is the bit error probability ϵ of a binary symmetric channel. Also shown for comparison is the channel's capacity $1 - H(\epsilon)$.

Proof: For each $\epsilon > 0$ and n sufficiently large

$$\frac{1}{2^{n/2}+2} < \epsilon$$

and, therefore, by Corollary IV.3, the indices of the nonempty cells are

$$\{i : 1 \le i \le 2^{n-1} - 1,\ i \text{ odd}\} \cup \{i : 2^{n-1} \le i \le 2^n - 2,\ i \text{ even}\}.$$

As n grows, the encoding cells $R_n(i)$ in Theorem IV.2 corresponding to $i = 2^{n-1} - 1, 2^{n-1}$ do not affect the cell density function. At the same time, the right endpoint of the encoding cell in Theorem IV.2 corresponding to $i = 1$ converges to ϵ and the left endpoint of the encoding cell in Theorem IV.2 corresponding to $i = 2^n - 2$ converges to $1 - \epsilon$. All other encoding cells have length $\delta 2^{1-n}$. Hence, in the limit as $n \to \infty$ they uniformly partition the interval $[\epsilon, 1 - \epsilon]$. □

For completeness, we derive the point density function of a DOUQ with the CNC. Analogous to the NBC, the cell density function in Theorem IV.5 is equal to the point density function for a sequence of DOUQs with the CNC.

Theorem IV.6: A sequence of DOUQs with the CNC index assignment has a point density function given by

$$\lambda(x) = \begin{cases} \frac{1}{1-2\epsilon}, & \text{for } \epsilon < x < 1 - \epsilon \\ 0, & \text{else.} \end{cases}$$

Proof: From [3], the codepoints of a DOUQ with the CNC index assignment satisfy

$$y_n(i) = \sum_{j=0}^{2^n-1} \left(\frac{j+1/2}{2^n}\right) p_n\left(\pi_n^{(\mathrm{CNC})}(j) | \pi_n^{(\mathrm{CNC})}(i)\right)$$
$$= 2^{-n} \cdot \begin{cases} (2^n-1)\epsilon + \delta(i+\epsilon) + \frac{1}{2}, & \text{for } i \text{ even} \\ (2^n-1)\epsilon + \delta(i-\epsilon) + \frac{1}{2}, & \text{for } i \text{ odd} \end{cases} \tag{11}$$

where (11) follows from Lemma III.3. Thus,

$$y_n(i+1) - y_n(i) = \begin{cases} 2^{-n}(1-4\epsilon^2), & \text{for } i \text{ odd} \\ 2^{-n}(1-2\epsilon)^2, & \text{for } i \text{ even} \end{cases}$$

which implies the codepoints are uniformly distributed in the limit as $n \to \infty$. Since

$$y_n(0) = \epsilon + 2^{-(n+1)}(1 - 4\epsilon^2) \to \epsilon$$

as $n \to \infty$, and

$$y_n(2^n - 1) = 1 - \epsilon - 2^{-(n+1)}(1 - 4\epsilon^2) \to 1 - \epsilon$$

as $n \to \infty$, the point density function is uniform on $(\epsilon, 1 - \epsilon)$. □

V. Distortion Analysis

Let $D_{EO}^{(\pi_n)}$ denote the end-to-end MSE of an EOUQ with index assignment π_n. Recall that

$$\Lambda = \{i : R_n(i) \ne \emptyset\}.$$

For $i \in \Lambda$, define the quantities

$$I_r(i) = \operatorname*{argmin}_{\substack{j \in \Lambda \\ c_n(j) > c_n(i)}} c_n(j)$$
$$I_l(i) = \operatorname*{argmax}_{\substack{j \in \Lambda \\ c_n(j) < c_n(i)}} c_n(j)$$
$$z_n(i) = \sup R_n(i)$$

(I_r and I_l are defined when the argmin and argmax, respectively, exist). Also, define

$$V = \{i : 1 \notin \overline{R}_n(i)\} \cap \Lambda$$
$$I_1 = V^c \cap \Lambda.$$

$I_r(i)$ and $I_l(i)$ are the indices of cells immediately to the right and left, respectively, of the cell with index i; V is the set of indices of nonempty cells that do not contain 1; and I_1 is the index of the nonempty cell containing 1.

Lemma V.1: The MSE of a EOUQ with index assignment π_n is

$$D_{EO}^{(\pi_n)} = \frac{1}{3} - 2^{-n-1} + 2^{-2n-2} + 2^{-n}\left[\sum_{i \in V} z_n^2(i) \cdot \alpha_n(i, I_r(i)) - \sum_{j=0}^{2^n-1} j p_n(\pi_n(j) | \pi_n(I_1))\right] + 2^{-2n} \sum_{j=0}^{2^n-1} (j + j^2) p_n(\pi_n(j) | \pi_n(I_1)).$$

The next two theorems give the MSEs for the NBC with a channel unoptimized uniform quantizer and with an EOUQ. Theorem V.2 was stated in [8] (see, e.g., [13] for a proof). The results are given as a function of the quantizer rate n and the channel bit error probability ϵ. Let $D_{CU}^{(\pi_n)}$ denote the end-to-end MSE of a channel unoptimized uniform quantizer with index assignment π_n.

With no channel noise, the MSE is $2^{-2n}/12$. If a quantizer with the NBC is designed for a noiseless channel but used on a noisy channel, then Theorem V.2 shows that (for large n) roughly $\epsilon/3$ is added to the MSE. If a quantizer with the NBC and a channel-optimized encoder is used on a noisy channel, then Theorem V.3 shows that (for large n) the MSE is reduced by roughly $\epsilon^2/3$ from the channel unoptimized case.

Theorem V.2: The MSE of a channel unoptimized uniform quantizer with the NBC index assignment is

$$D_{CU}^{(\mathrm{NBC})} = \frac{2^{-2n}}{12} + \frac{\epsilon}{3}(1-2^{-2n}).$$

Theorem V.3: The MSE of an EOUQ with the NBC index assignment is

$$D_{EO}^{(\mathrm{NBC})} = D_{CU}^{(\mathrm{NBC})} - \frac{\epsilon^2}{3}(1-2\epsilon)(1-2^{-n})(1-2^{-n+1}).$$

Proof: For the NBC

$$p_n(\pi_n(j)|\pi_n(i)) = p_n(j|i)$$

and Theorem III.1 and Corollary III.2 imply that

$$\begin{aligned} V &= \{0,1,\dots,2^n-2\} \\ I_r(i) &= i+1 \\ I_1 &= 2^n-1. \end{aligned}$$

Hence, Lemma V.1 gives

$$\begin{aligned} D_{EO}^{(\mathrm{NBC})} &- \left(\frac{1}{3} - 2^{-n-1} + 2^{-2n-2}\right) \\ &= 2^{-2n}\sum_{j=0}^{2^n-1} j^2 p_n(j|2^n-1) \\ &\quad + (2^{-2n}-2^{-n})\sum_{j=0}^{2^n-1} j p_n(j|2^n-1) \\ &\quad + 2^{-n}\sum_{i=0}^{2^n-2} z_n^2(i)\cdot\alpha_n(i,i+1) \\ &= \frac{\epsilon(4^n-1)+2\epsilon^2(2^n-1)(2^n-2)+3\delta(2^n-1)^2}{3\cdot 2^{2n}} \\ &\quad + (2^{-2n}-2^{-n})(2^n-1)(1-\epsilon) \\ &\quad - \delta 2^{-n}\sum_{i=0}^{2^n-2}[\epsilon+\delta(i+1)2^{-n}]^2 \qquad (12) \\ &= \frac{\epsilon(4^n-1)+2\epsilon^2(2^n-1)(2^n-2)-3\epsilon(2^n-1)^2}{3\cdot 2^{2n}} \\ &\quad - \delta 2^{-n}\Bigg[(2^n-1)\epsilon^2+(2^n-1)2\epsilon\delta 2^{-n} \\ &\quad + (2^n-1)(2^n-2)\epsilon\delta 2^{-n} \\ &\quad + \frac{(2^n-1)(2^n-2)(2^{n+1}-3)\delta^2 2^{-2n}}{6} \\ &\quad + (2^n-1)^2\delta^2 2^{-2n}\Bigg] \qquad (13) \end{aligned}$$

$$D_{EO}^{(\mathrm{NBC})} = \frac{2^{-2n}}{12} + \frac{(2^n-1)(2^n-2)(2\epsilon^3-\epsilon^2)+(2^{2n}-1)\epsilon}{3\cdot 2^{2n}} \qquad (14)$$

where the last three terms in (12) follow from Lemma II.3, Lemma II.2, as well as (6) and (10), respectively; and where (14) follows from (13) after some arithmetic. □

Let $D_{DO}^{(\pi_n)}$ denote the end-to-end MSE of a DOUQ with index assignment π_n. For a given n and ϵ, an index assignment $\pi_n \in S_n$ is said to be *optimal for an EOUQ* if for all $\pi_n' \in S_n$

$$D_{EO}^{(\pi_n)} \le D_{EO}^{(\pi_n')}$$

and is said to be *optimal for a DOUQ* if for all $\pi_n' \in S_n$

$$D_{DO}^{(\pi_n)} \le D_{DO}^{(\pi_n')}.$$

In [3], it was shown that for all n and all ϵ, the NBC is optimal for a DOUQ. Theorems V.2 and V.3 show that with the NBC, the reduction in MSE obtained by using a channel-optimized quantizer encoder instead of one obeying the nearest neighbor condition is

$$\frac{(\epsilon^2-2\epsilon^3)(2^n-1)(2^n-2)}{3\cdot 2^{2n}}.$$

The next two theorems show, however, that the NBC is not optimal for an EOUQ for all n and all ϵ.

Theorem V.4: The MSE of an EOUQ with the CNC index assignment is

$$D_{EO}^{(\mathrm{CNC})} = \begin{cases} D_1(n,\epsilon), & \text{for } 0 \le \epsilon < \epsilon_n^* \\ D_2(n,\epsilon), & \text{for } \epsilon_n^* \le \epsilon < \frac{1}{2^{n/2}+2} \\ D_3(n,\epsilon), & \text{for } \frac{1}{2^{n/2}+2} \le \epsilon < 1/2 \end{cases}$$

where

$$\begin{aligned} &D_1(n,\epsilon) \\ &= \frac{2^{-2n}}{3(1+2\epsilon)}\Big((1/4)+(2^{2n}+(5/2))\epsilon \\ &\quad - (2^{2n+1}-15\cdot 2^n+4)\epsilon^2 \\ &\quad + 6(2^{2n}-2^{n+2}-4)\epsilon^3 \\ &\quad + (2^n-4)(2^n-2)\epsilon^4 - 12(2^n-4)\epsilon^5\Big) \\ &D_2(n,\epsilon) \\ &= \frac{2^{-3n}}{3}\Big(2^n-3+[(2^n-3)(2^{2n}+10)-2^{n-1}+48]\epsilon \\ &\quad - [(2^n-6)(2^n-5)(2^n-4)-3(2^{2n}+2^{n+2}-48)]\epsilon^2 \\ &\quad + 2(2^n-4)(2^{2n}-11\cdot 2^n+6)\epsilon^3 \\ &\quad + 6(2^n-6)(2^n-4)\epsilon^4 + 24(2^n-4)\epsilon^5\Big) \\ &D_3(n,\epsilon) \\ &= \frac{2^{-3n}}{3}\Big(2^n+3+[(2^n-3)(2^{2n}+10)-2^{n-1}]\epsilon \\ &\quad - [(2^n-6)(2^n-5)(2^n-4)-3\cdot 2^{2n}]\epsilon^2 \\ &\quad + 2(2^n-6)(2^n-5)(2^n-4)\epsilon^3 \\ &\quad + 12(2^n-5)(2^n-4)\epsilon^4 + 24(2^n-4)\epsilon^5\Big). \end{aligned}$$

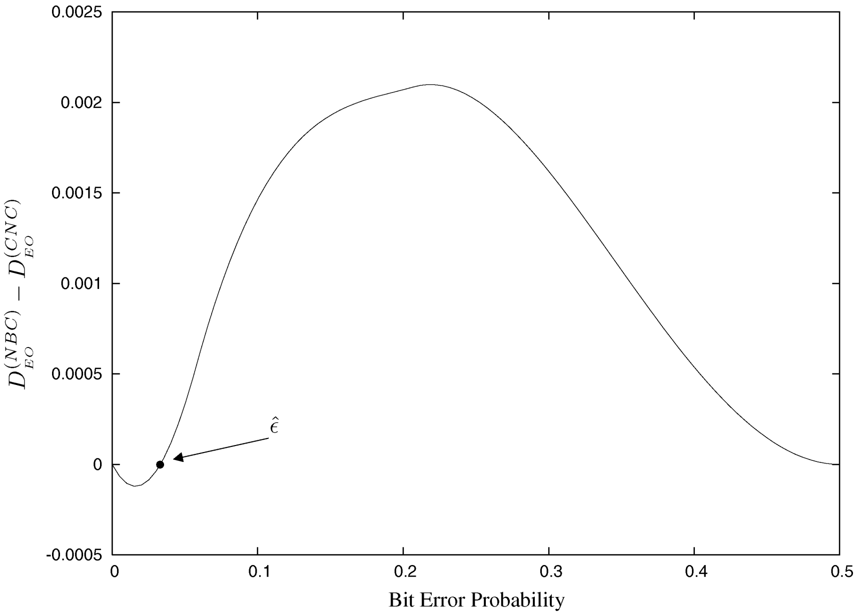

Fig. 6. Plot of the difference in MSE achieved by EOUQs with the NBC index assignment and CNC index assignment for rate $n = 3$. The horizontal axis is the bit error probability ϵ of a binary symmetric channel. The quantity $\hat{\epsilon}_n$ from Lemma V.5 is also shown.

Lemma V.5: On the interval $[0, 1/2]$, the polynomial

$$g_n(\epsilon) = 4(2^n - 4)\epsilon^4 + 2^n(2^n - 2)\epsilon^3 - 2(2^{2n} - 2^{n+2} - 4)\epsilon^2 + 2^n(2^n - 4)\epsilon - 1$$

has exactly one root $\hat{\epsilon}_n$, and $g_n(\epsilon) < 0$ if and only if $\epsilon < \hat{\epsilon}_n$. Furthermore

$$2^{-2n} < \hat{\epsilon}_n < 2^{-2n+1}$$

when $n \geq 4$.

Note that

$$g_n(2^{-2n+a}) \to 2^a - 1 > 0$$

as $n \to \infty$, for any $a > 0$. Hence, the bound on $\hat{\epsilon}_n$ can be strengthened to

$$2^{-2n} < \hat{\epsilon}_n < 2^{-2n+a}$$

for arbitrarily small $a > 0$ and sufficiently large n. Thus,

$$\hat{\epsilon}_n \sim 2^{-2n}$$

for asymptotically large n.

The following theorem shows that the quantity $\hat{\epsilon}_n$ defined in Lemma V.5 is a threshold value for the bit error probability of a binary symmetric channel, beyond which the MSE of an EOUQ with the CNC index assignment is smaller than with the NBC, for $n \geq 3$. Lemma V.5 then implies that the NBC is suboptimal for a large range of transmission rates and bit error probabilities (i.e., for all ϵ and n satisfying

$$\epsilon > 2^{-2n+o(1)}$$

where $o(1) \to 0$ as $n \to \infty$). In particular, for every $\epsilon > 0$, the CNC index assignment eventually outperforms the NBC for a large enough transmission rate. Fig. 6 plots the quantity

$$D_{EO}^{(\mathrm{NBC})} - D_{EO}^{(\mathrm{CNC})}$$

as a function of ϵ for rate $n = 3$.

Theorem V.6: $D_{EO}^{(\mathrm{CNC})} < D_{EO}^{(\mathrm{NBC})}$ if and only if $n \geq 3$ and $\epsilon > \hat{\epsilon}_n$.

Some intuition for why EOUQs with the CNC achieve lower MSEs than those with the NBC can be gained by examining the index generated by the CNC. For every $\epsilon > 0$ and for n sufficiently large, we have

$$\frac{1}{2^{n/2} + 2} < \epsilon$$

which, by Corollary IV.3, implies the indices of the nonempty cells in an EOUQ with the CNC are

$$\{i : 1 \leq i \leq 2^{n-1} - 1,\ i \text{ odd}\} \cup \{i : 2^{n-1} \leq i \leq 2^n - 2,\ i \text{ even}\}.$$

Corresponding to such nonempty cells, the encoder transmits (by the definition of CNC) only the odd integers $1, 3, \ldots, 2^n - 1$. Hence, the encoder of an EOUQ with the CNC emulates the encoder of a rate $n - 1$ EOUQ with the NBC, and then adds an extra bit (carrying no information) before transmission over the channel. Since the CNC uses longer codewords than the NBC, the CNC codewords are exposed to fewer channel errors on average, while being penalized with a lower level of quantizer resolution. This tradeoff makes the CNC superior to the NBC, except for very small bit error rates.

APPENDIX I
LEMMAS AND PROOFS FOR SECTION II

Lemma I.1: For any index assignment $\pi_n \in S_n$ and for $0 \le i \le 2^n - 1$

$$\sum_{j=0}^{2^n-1} (1-\epsilon)^{n-H_n(\pi_n(i),\pi_n(j))}\epsilon^{H_n(\pi_n(i),\pi_n(j))} = 1.$$

Proof of Lemma I.1: It follows immediately since index assignments are permutations. □

Proof of Lemma II.1: Let i and k be two distinct integers between 0 and $2^n - 1$. Then the inequality in (2) can be rewritten as

$$\sum_{j=0}^{2^n-1} [x^2 - 2xy_n(j) + y_n^2(j)] p_n(\pi_n(j)|\pi_n(i)) \le \sum_{j=0}^{2^n-1} [x^2 - 2xy_n(j) + y_n^2(j)] p_n(\pi_n(j)|\pi_n(k)).$$

Since π_n is bijective and $\sum_j p_n(j|i) = 1, \ \forall j$, cancellation of terms gives

$$\sum_{j=0}^{2^n-1} \left[\frac{-2xj}{2^n} + \frac{(j^2+j)}{2^{2n}}\right] p_n(\pi_n(j)|\pi_n(i)) \le \sum_{j=0}^{2^n-1} \left[\frac{-2xj}{2^n} + \frac{(j^2+j)}{2^{2n}}\right] p_n(\pi_n(j)|\pi_n(k))$$

or equivalently

$$x\sum_{j=0}^{2^n-1} j[p_n(\pi_n(j)|\pi_n(i)) - p_n(\pi_n(j)|\pi_n(k))] \ge 2^{-n-1}\sum_{j=0}^{2^n-1} (j^2+j)[p_n(\pi_n(j)|\pi_n(i)) - p_n(\pi_n(j)|\pi_n(k))].$$ □

APPENDIX II
LEMMAS AND PROOFS FOR SECTION III

The following lemma is easy to prove and is used in the proofs of Lemmas II.2 and II.3.

Lemma II.1:

$$H_{n+1}(i,j) = H_n(i,j), \quad \text{for } 0 \le i,j \le 2^n - 1 \tag{B1}$$

$$H_{n+1}(i,j+2^n) = H_n(i,j) + 1, \quad \text{for } 0 \le i,j \le 2^n - 1 \tag{B2}$$

$$H_{n+1}(i,j) = H_n(i-2^n,j) + 1, \quad \text{for } 0 \le j \le 2^n - 1 \text{ and } 2^n \le i \le 2^{n+1} - 1 \tag{B3}$$

$$H_{n+1}(i,j) = H_n(i-2^n,j-2^n), \quad \text{for } 2^n \le i,j \le 2^{n+1} - 1. \tag{B4}$$

Lemma II.2: If $0 \le i \le 2^n - 1$, then

$$\sum_{j=0}^{2^n-1} jp_n\left(\pi_n^{(\mathrm{NBC})}(j)|\pi_n^{(\mathrm{NBC})}(i)\right) = (2^n-1)\epsilon + i\delta. \tag{B5}$$

Proof of Lemma II.2: We use induction on n. The case of $n = 1$ is true since

$$p_1\left(\pi_1^{(\mathrm{NBC})}(j)|\pi_1^{(\mathrm{NBC})}(i)\right) = p_1(j|i)$$
$$\sum_{j=0}^{1} jp_1(j|0) = \epsilon$$
$$\sum_{j=0}^{1} jp_1(j|1) = 1 - \epsilon.$$

Now assume (B5) is true for n and consider two cases for $n+1$.

If $0 \le i \le 2^n - 1$, then using (B1) and (B2) to express $p_{n+1}(j|i)$ in terms of $p_n(j|i)$ and simplifying with Lemma I.1 gives

$$\begin{aligned}\sum_{j=0}^{2^{n+1}-1} jp_{n+1}(j|i) &= (1-\epsilon)\sum_{j=0}^{2^n-1} jp_n(j|i) + \epsilon\sum_{j=0}^{2^n-1} jp_n(j|i) + 2^n\epsilon \\ &= (2^{n+1}-1)\epsilon + i\delta \end{aligned} \tag{B6}$$

where (B6) follows from the induction hypothesis.

If $2^n \le i \le 2^{n+1} - 1$, then using (B3) and (B4) to express $p_{n+1}(j|i)$ in terms of $p_n(j|i)$ and simplifying with Lemma I.1 gives

$$\begin{aligned}\sum_{j=0}^{2^{n+1}-1} jp_{n+1}(j|i) &= \epsilon\sum_{j=0}^{2^n-1} jp_n(j|i-2^n) \\ &\quad + (1-\epsilon)\sum_{j=0}^{2^n-1} jp_n(j|i-2^n) + 2^n(1-\epsilon) \\ &= (2^{n+1}-1)\epsilon + i\delta \end{aligned} \tag{B7}$$

where (B7) follows from the induction hypothesis. □

Lemma II.3: If $0 \le i \le 2^n - 1$, then

$$\begin{aligned}\sum_{j=0}^{2^n-1} j^2p_n\left(\pi_n^{(\mathrm{NBC})}(j)|\pi_n^{(\mathrm{NBC})}(i)\right) &= \frac{\epsilon}{3}(4^n-1) + \frac{2\epsilon^2}{3}(2^n-1)(2^n-2) \\ &\quad + i2\epsilon\delta(2^n-1) + i^2\delta^2. \end{aligned} \tag{B8}$$

Proof of Lemma II.3: We use induction on n. The case of $n = 1$ is true since

$$p_1\left(\pi_1^{(\mathrm{NBC})}(j)|\pi_1^{(\mathrm{NBC})}(i)\right) = p_1(j|i)$$
$$\sum_{j=0}^{1} j^2p_1(j|0) = \epsilon$$
$$\sum_{j=0}^{1} j^2p_1(j|1) = 1 - \epsilon$$

which satisfies the right-hand side of (B8). Now assume (B8) is true for n and consider two cases for $n+1$.

If $0 \le i \le 2^n - 1$, then using (B1) and (B2) to express $p_{n+1}(j|i)$ in terms of $p_n(j|i)$ and simplifying with Lemma I.1 gives

$$\begin{aligned}\sum_{j=0}^{2^{n+1}-1} j^2 p_{n+1}(j|i) &= (1-\epsilon)\sum_{j=0}^{2^n-1} j^2 p_n(j|i) + \epsilon \sum_{j=0}^{2^n-1} j^2 p_n(j|i) \\ &\quad + \epsilon\, 2^{n+1}\sum_{j=0}^{2^n-1} j p_n(j|i) + 2^{2n}\epsilon \\ &= \frac{\epsilon}{3}(4^{n+1}-1) + \frac{2\epsilon^2}{3}(2^{n+1}-1)(2^{n+1}-2) \\ &\quad + i2\epsilon\delta(2^{n+1}-1) + i^2\delta^2 \qquad \text{(B9)}\end{aligned}$$

where (B9) follows from the induction hypothesis and Lemma II.2.

If $2^n \le i \le 2^{n+1}-1$, then using (B3) and (B4) to express $p_{n+1}(j|i)$ in terms of $p_n(j|i)$ and simplifying with Lemma I.1 gives

$$\begin{aligned}\sum_{j=0}^{2^{n+1}-1} j^2 p_{n+1}(j|i) &= \epsilon\sum_{j=0}^{2^n-1} j^2 p_n(j|i-2^n) \\ &\quad + (1-\epsilon)\sum_{j=0}^{2^n-1} j^2 p_n(j|i-2^n) \\ &\quad + (1-\epsilon)2^{n+1}\sum_{j=0}^{2^n-1} j p_n(j|i-2^n) + 2^{2n}(1-\epsilon) \\ &= \frac{\epsilon}{3}(4^{n+1}-1) + \frac{2\epsilon^2}{3}(2^{n+1}-1)(2^{n+1}-2) \\ &\quad + i2\epsilon\delta(2^{n+1}-1) + i^2\delta^2 \qquad \text{(B10)}\end{aligned}$$

where (B10) follows from the induction hypothesis and Lemma II.2. □

APPENDIX III
LEMMAS AND PROOFS FOR SECTION IV

The following two lemmas are used in the proofs of Lemmas III.3 and III.5. Let

$$\hat{H}_n(i,j) = H(\pi_n^{(\mathrm{CNC})}(i), \pi_n^{(\mathrm{CNC})}(j)).$$

Lemma III.1:

$$\begin{aligned}\hat{H}_n(i,j) &= H_n(i,j), \\ &\quad \text{for } 0 \le i,j \le 2^{n-1}-1 \\ &\quad \text{or } 2^{n-1} \le i,j \le 2^n - 1 \qquad \text{(C1)} \\ \hat{H}_n(i,j) &= H_n(i,j+1), \\ &\quad \text{for } 0 \le i \le 2^{n-1}-1,\ 2^{n-1} \le j \le 2^n-2, \\ &\quad \text{and } j \text{ even} \qquad \text{(C2)} \\ \hat{H}_n(i,j) &= H_n(i,j-1), \\ &\quad \text{for } 0 \le i \le 2^{n-1}-1,\ 2^{n-1}+1 \le j \le 2^n-1, \\ &\quad \text{and } j \text{ odd} \qquad \text{(C3)} \\ \hat{H}_n(i,j) &= H_n(i,j+1), \\ &\quad \text{for } 2^{n-1} \le i \le 2^n-1,\ 0 \le j \le 2^{n-1}-2, \\ &\quad \text{and } j \text{ even} \qquad \text{(C4)} \\ \hat{H}_n(i,j) &= H_n(i,j-1), \\ &\quad \text{for } 2^{n-1} \le i \le 2^n-1,\ 1 \le j \le 2^{n-1}-1, \\ &\quad \text{and } j \text{ odd.} \qquad \text{(C5)}\end{aligned}$$

Proof of Lemma III.1: It follows from the definition of the CNC. □

Lemma III.2: If $0 \le i \le 2^{n-1}-1$, then

$$\sum_{\substack{j=2^{n-1} \\ j \text{ even}}}^{2^n-2} (1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} = \begin{cases}\epsilon(1-\epsilon), & \text{for } i \text{ even} \\ \epsilon^2, & \text{for } i \text{ odd}\end{cases} \qquad \text{(C6)}$$

$$\sum_{\substack{j=2^{n-1}+1 \\ j \text{ odd}}}^{2^n-1} (1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} = \begin{cases}\epsilon^2, & \text{for } i \text{ even} \\ \epsilon(1-\epsilon), & \text{for } i \text{ odd}\end{cases} \qquad \text{(C7)}$$

and if $2^{n-1} \le i \le 2^n - 1$, then

$$\sum_{\substack{j=0 \\ j \text{ even}}}^{2^{n-1}-2} (1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} = \begin{cases}\epsilon(1-\epsilon), & \text{for } i \text{ even} \\ \epsilon^2, & \text{for } i \text{ odd}\end{cases} \qquad \text{(C8)}$$

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{2^{n-1}-1} (1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} = \begin{cases}\epsilon^2, & \text{for } i \text{ even} \\ \epsilon(1-\epsilon), & \text{for } i \text{ odd.}\end{cases} \qquad \text{(C9)}$$

Proof of Lemma III.2: It follows from the definition of the NBC. □

Lemma III.3: If $0 \le i \le 2^n - 1$, then

$$\sum_{j=0}^{2^n-1} j p_n\left(\pi_n^{(CNC)}(j)|\pi_n^{(CNC)}(i)\right) = \begin{cases}(2^n-1)\epsilon + \delta(i+\epsilon), & \text{for } i \text{ even} \\ (2^n-1)\epsilon + \delta(i-\epsilon), & \text{for } i \text{ odd.}\end{cases}$$

Proof of Lemma III.3: If $0 \le i \le 2^{n-1}-1$, then using (C1)–(C3) in Lemma III.1 to express

$$p_n\left(\pi_n^{(\mathrm{CNC})}(j)|\pi_n^{(\mathrm{CNC})}(i)\right)$$

in terms of ϵ, n, and $H_n(i,j)$ gives

$$\sum_{j=0}^{2^n-1} j p_n\left(\pi_n^{(\mathrm{CNC})}(j)|\pi_n^{(\mathrm{CNC})}(i)\right)$$

$$= \sum_{j=0}^{2^{n-1}-1} j(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} + \sum_{\substack{j=2^{n-1}+1 \\ j \text{ odd}}}^{2^n-1} (j-1)(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} + \sum_{\substack{j=2^{n-1} \\ j \text{ even}}}^{2^n-2} (j+1)(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)}$$

$$= \begin{cases} (2^n-1)\epsilon + \delta(i+\epsilon), & \text{for } i \text{ even} \\ (2^n-1)\epsilon + \delta(i-\epsilon), & \text{for } i \text{ odd} \end{cases} \tag{C10}$$

where (C10) follows from Lemma II.2 and (C6) and (C7) in Lemma III.2.

If $2^{n-1} \le i \le 2^n - 1$ then using (C4), (C5), and (C1) in Lemma III.1 to express

$$p_n\left(\pi_n^{(\text{CNC})}(j)|\pi_n^{(\text{CNC})}(i)\right)$$

in terms of ϵ, n, and $H_n(i,j)$ gives

$$\sum_{j=0}^{2^n-1} jp_n\left(\pi_n^{(\text{CNC})}(j)|\pi_n^{(\text{CNC})}(i)\right) = \sum_{\substack{j=1 \\ j \text{ odd}}}^{2^{n-1}-1} (j-1)(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} + \sum_{\substack{j=0 \\ j \text{ even}}}^{2^{n-1}-2} (j+1)(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} + \sum_{j=2^{n-1}}^{2^n-1} j(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)}$$

$$= \begin{cases} (2^n-1)\epsilon + \delta(i+\epsilon), & \text{for } i \text{ even} \\ (2^n-1)\epsilon + \delta(i-\epsilon), & \text{for } i \text{ odd} \end{cases} \tag{C11}$$

where (C11) follows from Lemma II.2 and (C8) and (C9) in Lemma III.2. □

The following lemma is used in the proof of Lemma III.5.

Lemma III.4: If $0 \le i \le 2^{n-1} - 1$, then

$$\sum_{\substack{j=2^{n-1} \\ j \text{ even}}}^{2^n-2} j(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} = \begin{cases} 2^{n-1}\epsilon(1-\epsilon^2) - 2\epsilon^2(1-\epsilon) \\ \quad + i\delta(1-\epsilon)\epsilon, & \text{for } i \text{ even} \\ (2^{n-1}-1)\epsilon^2 + i\delta\epsilon^2 + 2^{n-1}\epsilon^3, & \text{for } i \text{ odd} \end{cases} \tag{C12}$$

$$\sum_{\substack{j=2^{n-1}+1 \\ j \text{ odd}}}^{2^n-1} j(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} = \begin{cases} (2^{n-1}+1)\epsilon^2 + i\delta\epsilon^2 \\ \quad +(2^{n-1}-2)\epsilon^3, & \text{for } i \text{ even} \\ 2^{n-1}\epsilon(1-\epsilon^2) + i\delta(1-\epsilon)\epsilon, & \text{for } i \text{ odd} \end{cases} \tag{C13}$$

and if $2^{n-1} \le i \le 2^n - 1$, then

$$\sum_{\substack{j=0 \\ j \text{ even}}}^{2^{n-1}-2} j(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} = \begin{cases} (2^{n-1}-1)\epsilon^2(1-\epsilon) - \epsilon^2(1-\epsilon) \\ \quad + (i-2^{n-1})\delta(1-\epsilon)\epsilon, & \text{for } i \text{ even} \\ (2^{n-1}-1)\epsilon^3 - \epsilon^2(1-\epsilon) \\ \quad + (i-2^{n-1})\delta\epsilon^2, & \text{for } i \text{ odd} \end{cases} \tag{C14}$$

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{2^{n-1}-1} j(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} = \begin{cases} (2^{n-1}-1)\epsilon^3 + \epsilon^2(1-\epsilon) \\ \quad + (i-2^{n-1})\delta\epsilon^2, & \text{for } i \text{ even} \\ (2^{n-1}-1)\epsilon^2(1-\epsilon) + \epsilon^2(1-\epsilon) \\ \quad + (i-2^{n-1})\delta(1-\epsilon)\epsilon, & \text{for } i \text{ odd.} \end{cases} \tag{C15}$$

Proof of Lemma III.4: For each sum in (C12)–(C15), the first and last digits of the binary expansions of i and j are constant over all terms in the sum. Therefore, their contribution to the Hamming distance $H_n(i,j)$ is the same for each term in the sum. Hence, by summing over the middle $n-2$ bits of j, the left-hand sides of (C12)–(C15) are $(1-\epsilon)^n$ times, respectively,

$$\begin{cases} \sum_{j=0}^{2^{n-2}-1} (2j+2^{n-1}) \left(\frac{\epsilon}{1-\epsilon}\right)^{H_{n-2}(i/2,j)+1}, & \text{for } i \text{ even} \\ \sum_{j=0}^{2^{n-2}-1} (2j+2^{n-1}) \left(\frac{\epsilon}{1-\epsilon}\right)^{H_{n-2}((i-1)/2,j)+2}, & \text{for } i \text{ odd,} \end{cases}$$

$$\begin{cases} \sum_{j=0}^{2^{n-2}-1} (2j+1+2^{n-1}) \left(\frac{\epsilon}{1-\epsilon}\right)^{H_{n-2}(i/2,j)+2}, & \text{for } i \text{ even} \\ \sum_{j=0}^{2^{n-2}-1} (2j+1+2^{n-1}) \left(\frac{\epsilon}{1-\epsilon}\right)^{H_{n-2}((i-1)/2,j)+1}, & \text{for } i \text{ odd,} \end{cases}$$

$$\begin{cases} \sum_{j=0}^{2^{n-2}-1} 2j \left(\frac{\epsilon}{1-\epsilon}\right)^{H_{n-2}((i-2^{n-1})/2,j)+1}, & \text{for } i \text{ even} \\ \sum_{j=0}^{2^{n-2}-1} 2j \left(\frac{\epsilon}{1-\epsilon}\right)^{H_{n-2}((i-1-2^{n-1})/2,j)+2}, & \text{for } i \text{ odd,} \end{cases}$$

$$\begin{cases} \sum_{j=0}^{2^{n-2}-1} (2j+1) \left(\frac{\epsilon}{1-\epsilon}\right)^{H_{n-2}((i-2^{n-1})/2,j)+2}, & \text{for } i \text{ even} \\ \sum_{j=0}^{2^{n-2}-1} (2j+1) \left(\frac{\epsilon}{1-\epsilon}\right)^{H_{n-2}((i-1-2^{n-1})/2,j)+1}, & \text{for } i \text{ odd.} \end{cases}$$

The right-hand sides of (C12)–(C15) then follow from Lemma II.2. □

Lemma III.5: If $0 \le i \le 2^{n-1} - 1$, then

$$\sum_{j=0}^{2^n-1} j^2 p_n\left(\pi_n^{(\text{CNC})}(j)|\pi_n^{(\text{CNC})}(i)\right) = \begin{cases} \epsilon\left[\left(\frac{2^{2n}-1}{3}\right)+2^n+1\right] + \frac{\epsilon^2}{3}(2^{2n+1}-9\cdot 2^n-14) \\ \quad -\epsilon^3(2^{n+1}-8) + i\delta[2\epsilon(2^n-1)+2\epsilon\delta] + i^2\delta^2, & \text{for } i \text{ even} \\ \epsilon\left[\left(\frac{2^{2n}-1}{3}\right)-2^n+1\right] + \frac{\epsilon^2}{3}(2^{2n+1}-3\cdot 2^n-2) \\ \quad +2^{n+1}\epsilon^3 + i\delta[2\epsilon(2^n-1)-2\epsilon\delta] + i^2\delta^2, & \text{for } i \text{ odd} \end{cases}$$

and if $2^{n-1} \leq i \leq 2^n - 1$, then

$$\sum_{j=0}^{2^n-1} j^2 p_n\Big(\pi_n^{(\mathrm{CNC})}(j)|\pi_n^{(\mathrm{CNC})}(i)\Big) = \begin{cases} \epsilon\left[\left(\frac{2^{2n}-1}{3}\right)-2^n+1\right]+\frac{\epsilon^2}{3}(2^{2n+1}+9\cdot 2^n-14) \\ -\epsilon^3(3\cdot 2^{n+1}-8)+i\delta[2\epsilon(2^n-1)+2\epsilon\delta]+i^2\delta^2, & \text{for } i \text{ even} \\ \epsilon\left[\left(\frac{2^{2n}-1}{3}\right)+2^n+1\right]+\frac{\epsilon^2}{3}(2^{2n+1}-21\cdot 2^n-2) \\ +3\cdot 2^{n+1}\epsilon^3+i\delta[2\epsilon(2^n-1)-2\epsilon\delta]+i^2\delta^2, & \text{for } i \text{ odd.} \end{cases}$$

Proof of Lemma III.5: If $0 \leq i \leq 2^{n-1} - 1$, then using (C1)–(C3) in Lemma III.1 to express

$$p_n\left(\pi_n^{(\mathrm{CNC})}(j)|\pi_n^{(\mathrm{CNC})}(i)\right)$$

in terms of ϵ, n, and $H_n(i,j)$, and simplifying with (C6) and (C7) from Lemma III.2 gives

$$\begin{aligned}\sum_{j=0}^{2^n-1} j^2 p_n\Big(\pi_n^{(\mathrm{CNC})}(j)|\pi_n^{(\mathrm{CNC})}(i)\Big) &= \sum_{j=0}^{2^n-1} j^2(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} + \epsilon \\ &\quad - 2\left[\sum_{\substack{j=2^{n-1}+1 \\ j \text{ odd}}}^{2^n-1} j(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} - \sum_{\substack{j=2^{n-1} \\ j \text{ even}}}^{2^n-2} j(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)}\right] \\ &= \begin{cases} \epsilon\left[\left(\frac{2^{2n}-1}{3}\right)+2^n+1\right]+\frac{\epsilon^2}{3}(2^{2n+1}-9\cdot 2^n-14) \\ -\epsilon^3(2^{n+1}-8)+i\delta[2\epsilon(2^n-1)+2\epsilon\delta]+i^2\delta^2, & \text{for } i \text{ even} \\ \epsilon\left[\left(\frac{2^{2n}-1}{3}\right)-2^n+1\right]+\frac{\epsilon^2}{3}(2^{2n+1}-3\cdot 2^n-2) \\ +2^{n+1}\epsilon^3+i\delta[2\epsilon(2^n-1)-2\epsilon\delta]+i^2\delta^2, & \text{for } i \text{ odd} \end{cases}\end{aligned} \tag{C16}$$

where (C16) follows from Lemma II.3 and (C12) and (C13) in Lemma III.4.

If $2^{n-1} \leq i \leq 2^n - 1$, then using (C1), (C4), and (C5) in Lemma III.1 to express

$$p_n\left(\pi_n^{(\mathrm{CNC})}(j)|\pi_n^{(\mathrm{CNC})}(i)\right)$$

in terms of ϵ, n, and $H_n(i,j)$, and simplifying with (C8) and (C9) from Lemma III.2 gives

$$\begin{aligned}\sum_{j=0}^{2^n-1} j^2 p_n\left(\pi_n^{(\mathrm{CNC})}(j)|\pi_n^{(\mathrm{CNC})}(i)\right) &= \sum_{j=0}^{2^n-1} j^2(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} + \epsilon \\ &\quad - 2\left[\sum_{\substack{j=1 \\ j \text{ odd}}}^{2^{n-1}-1} j(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)} - \sum_{\substack{j=0 \\ j \text{ even}}}^{2^{n-1}-2} j(1-\epsilon)^{n-H_n(i,j)}\epsilon^{H_n(i,j)}\right] \\ &= \begin{cases} \epsilon\left[\left(\frac{2^{2n}-1}{3}\right)-2^n+1\right]+\frac{\epsilon^2}{3}(2^{2n+1}+9\cdot 2^n-14) \\ -\epsilon^3(3\cdot 2^{n+1}-8)+i\delta[2\epsilon(2^n-1)+2\epsilon\delta]+i^2\delta^2, & \text{for } i \text{ even} \\ \epsilon\left[\left(\frac{2^{2n}-1}{3}\right)+2^n+1\right]+\frac{\epsilon^2}{3}(2^{2n+1}-21\cdot 2^n-2) \\ +3\cdot 2^{n+1}\epsilon^3+i\delta[2\epsilon(2^n-1)-2\epsilon\delta]+i^2\delta^2, & \text{for } i \text{ odd} \end{cases}\end{aligned} \tag{C17}$$

where (C17) follows from Lemma II.3 and (C14) and (C15) in Lemma III.4. □

Proof of Lemma IV.1: Since

$$\begin{aligned}\phi_n''' &< 0 \\ \phi_n(0) &= -1 \\ \phi_n(1/2) &= 2^{n-1} \\ \phi_n(1) &= -3\end{aligned}$$

the cubic function ϕ_n has exactly one root (i.e., ϵ_n^*) in $(0, 1/2)$, $\phi_n < 0$ on $[0, \epsilon_n^*)$, and $\phi_n > 0$ on $(\epsilon_n^*, 1/2]$. Furthermore, $\epsilon_n^* > \epsilon_{n+1}^*$ since

$$\phi_{n+1}(\epsilon) - \phi_n(\epsilon) = 2^{n+1}\epsilon(1-\epsilon), \qquad \forall \epsilon.$$

The fact that

$$\epsilon_n^* < \frac{1}{2^{n/2}+2}$$

follows from the fact that

$$\phi_n\left(\frac{1}{2^{n/2}+2}\right) = \frac{2^{2n+1}+5\cdot 2^{3n/2}}{(2^{n/2}+2)^3} > 0. \qquad \square$$

Proof of Theorem IV.2: The encoding cells satisfy (3) in Lemma II.1, with

$$\begin{aligned}\alpha_n(i,k) &= \sum_{j=0}^{2^n-1} j\left[p_n(\pi_n^{(\mathrm{CNC})}(j)|\pi_n^{(\mathrm{CNC})}(i)) - p_n(\pi_n^{(\mathrm{CNC})}(j)|\pi_n^{(\mathrm{CNC})}(k))\right] \\ &= \begin{cases} \delta(i-k), & \text{for } i,k \text{ even or } i,k \text{ odd} \\ \delta(i-k-2\epsilon), & \text{for } k \text{ even, } i \text{ odd} \\ \delta(i-k+2\epsilon), & \text{for } i \text{ even, } k \text{ odd} \end{cases}\end{aligned} \tag{C18}$$

where (C18) follows from Lemma III.3. Let

$$\rho_n(i,k) = \beta_n(i,k)/\alpha_n(i,k)$$

for $i \neq k$. Note that $\rho_n(i,k)$ is well defined because $\alpha_n(i,k) \neq 0$ whenever $k \neq i$, by (C18). Also, from (C18), we have that $\alpha_n(i,k) > 0$ if and only if $i > k$, and $\alpha_n(i,k) < 0$ if and only if $i < k$.

Thus, (3) can be rewritten as

$$\begin{aligned}\overline{R}_n(i) &= \{x \in [0,1] : x \geq \rho_n(i,k),\ \forall\, k < i \\ &\qquad \text{and} \quad x \leq \rho_n(i,k),\ \forall\, k > i\} \\ &= \left\{x \in [0,1] : \max_{k<i} \rho_n(i,k) \leq x \leq \min_{k>i} \rho_n(i,k)\right\}.\end{aligned} \tag{C19}$$

Therefore, the encoding cell with index i is empty if and only if at least one of the following conditions holds:

$$\max_{k<i} \rho_n(i,k) \geq \min_{k>i} \rho_n(i,k) \quad \text{(C20)}$$

$$\min_{k>i} \rho_n(i,k) \leq 0 \quad \text{(C21)}$$

$$\max_{k<i} \rho_n(i,k) \geq 1. \quad \text{(C22)}$$

For notational convenience, assume

$$\max_{k<i} \rho_n(0,k) = 0$$

$$\min_{k>i} \rho_n(2^n-1,k) = 1.$$

We will examine four cases, corresponding to the parity and size of a cell's index i.

Case 1: i even, $0 \leq i \leq 2^{n-1}-2$

Equations (5) and (C18) as well as Lemma III.5 imply (C23) at the bottom of the page. Equations (C24) and (C25) follow from (C23) and the fact that $\rho_n(i,k)$ is increasing in k for $k < i$ and $k > i$.

$$\max_{\substack{k<i\\ i\neq 0}} \rho_n(i,k) = \left(i\delta + \frac{2^n\epsilon(2+\epsilon)}{1+2\epsilon}\right)2^{-n} \quad \text{(C24)}$$

$$\min_{k>i} \rho_n(i,k) = \left((i+1)\delta - \frac{2^n\epsilon^2}{\delta}\right)2^{-n}. \quad \text{(C25)}$$

For $i \neq 0$, the ith encoding cell $R_n(i)$ is nonempty if and only if the conditions in (C20)–(C22) are each false. Equations (C24), (C25) and Lemma IV.1 imply (C20) is false if and only if

$$\left(i\delta + \frac{2^n\epsilon(2+\epsilon)}{1+2\epsilon}\right)2^{-n} < \left((i+1)\delta - \frac{2^n\epsilon^2}{\delta}\right)2^{-n}$$

or, equivalently, if and only if

$$\epsilon < \epsilon_n^*. \quad \text{(C26)}$$

Equation (C21) is false if and only if (C25) is positive, or equivalently

$$\epsilon < \frac{1}{2^{(n/2)-(1/2)\log(i+1)}+2}. \quad \text{(C27)}$$

Similarly, (C22) is false if and only if (C24) is less than 1, or equivalently

$$i < \frac{2^n(1-\epsilon^2)}{1-4\epsilon^2} \quad \text{(C28)}$$

which is always true, since

$$\frac{1-\epsilon^2}{1-4\epsilon^2} > 1$$

and $i < 2^n$. Lemma IV.1 implies that

$$\epsilon_n^* < \frac{1}{2^{(n/2)-(1/2)\log(i+1)}+2}$$

for $i \geq 0$. Hence, if $\epsilon < \epsilon_n^*$, then (C27) holds, and therefore, $R_n(i)$ is nonempty for $i \neq 0$ if and only if $\epsilon < \epsilon_n^*$.

For $i = 0$ the conditions in (C20) and (C21) are equivalent and the condition in (C22) is always false. Therefore, the encoding cell $R_n(0)$ is nonempty (from (C21) and (C25)) if and only if

$$\epsilon < \frac{1}{2^{n/2}+2}. \quad \text{(C29)}$$

Case 2: i odd, $1 \leq i \leq 2^{n-1}-1$

Equations (5) and (C18) as well as Lemma III.5 imply (C30) at the bottom of the page.

If $i \neq 1$, then from (C30)

$$\begin{aligned}\max_{k<i} \rho_n(i,k) &= \max\left\{ \max_{\substack{k<i\\ k\ \text{even}}} \epsilon + 2^{-n-1}(i+k+1)\delta - \frac{\epsilon(1-\epsilon)}{i-k-2\epsilon},\right.\\ &\qquad\left. \epsilon + 2^{-n-1}(i+k+\delta)\delta\Big|_{k=i-2}\right\} \quad \text{(C31)}\\ &= \begin{cases} \left(i\delta - \frac{2^n\epsilon^2}{\delta}\right)2^{-n}, & \text{for } \epsilon < \epsilon_n^* \\ \left((2i-2)\delta + \delta^2 + 2^{n+1}\epsilon\right)2^{-n-1}, & \text{for } \epsilon \geq \epsilon_n^*. \end{cases} \quad \text{(C32)}\end{aligned}$$

Equation (C32) was obtained by noting that in (C31) the first term is greater than the second term if and only if both $k = i-1$ (since k is even) and (after some algebra) $\phi(\epsilon) < 0$ (i.e., $\epsilon < \epsilon_n^*$ via Lemma IV.1). If $i = 1$, then from (C30)

$$\max_{k<1} \rho_n(i,k) = \rho_n(1,0) = \left(\delta - \frac{2^n\epsilon^2}{\delta}\right)2^{-n}. \quad \text{(C33)}$$

$$\rho_n(i,k) = \begin{cases} \epsilon + 2^{-n-1}(i+k+1+2\epsilon)\delta, & \text{for } 0 \leq k \leq 2^{n-1}-2,\ k \text{ even} \\ \epsilon + 2^{-n-1}(i+k+1+2\epsilon)\delta + \frac{\epsilon(1-\epsilon)}{i-k}, & \text{for } 2^{n-1} \leq k \leq 2^n-2,\ k \text{ even} \\ \epsilon + 2^{-n-1}(i+k+1)\delta + \frac{\epsilon(1-\epsilon)}{i-k+2\epsilon}, & \text{for } 1 \leq k \leq 2^{n-1}-1,\ k \text{ odd} \\ \epsilon + 2^{-n-1}(i+k+1)\delta, & \text{for } 2^{n-1}+1 \leq k \leq 2^n-1,\ k \text{ odd.} \end{cases} \quad \text{(C23)}$$

$$\rho_n(i,k) = \begin{cases} \epsilon + 2^{-n-1}(i+k+1)\delta - \frac{\epsilon(1-\epsilon)}{i-k-2\epsilon}, & \text{for } 0 \leq k \leq 2^{n-1}-2,\ k \text{ even} \\ \epsilon + 2^{-n-1}(i+k+1)\delta, & \text{for } 2^{n-1} \leq k \leq 2^n-2,\ k \text{ even} \\ \epsilon + 2^{-n-1}(i+k+\delta)\delta, & \text{for } 1 \leq k \leq 2^{n-1}-1,\ k \text{ odd} \\ \epsilon + 2^{-n-1}(i+k+\delta)\delta - \frac{\epsilon(1-\epsilon)}{i-k}, & \text{for } 2^{n-1}+1 \leq k \leq 2^n-1,\ k \text{ odd.} \end{cases} \quad \text{(C30)}$$

For $i \neq 2^{n-1}-1$

$$\begin{aligned}\min_{k>i} \rho_n(i,k) &= \min \Bigg\{ \min_{\substack{k>i \\ 2\le k<2^{n-1} \\ k\ even}} \epsilon + 2^{-n-1}(i+k+1)\delta - \frac{\epsilon(1-\epsilon)}{i-k-2\epsilon}, \\ &\quad \epsilon + 2^{-n-1}(i+k+1)\delta\Big|_{k=2^{n-1}}, \\ &\quad \epsilon + 2^{-n-1}(i+k+\delta)\delta\Big|_{k=i+2}, \\ &\quad \min_{\substack{i<k \\ 2^{n-1}+1<k<2^n-1 \\ k\ odd}} \epsilon + 2^{-n-1}(i+k+\delta)\delta - \frac{\epsilon(1-\epsilon)}{i-k} \Bigg\} \end{aligned} \tag{C34}$$

$$= \begin{cases} \left((i+1)\delta + \frac{2^n\epsilon(2+\epsilon)}{1+2\epsilon}\right)2^{-n}, & \text{for } \epsilon < \epsilon_n^* \\ \left((2i+2)\delta + \delta^2 + 2^{n+1}\epsilon\right)2^{-n-1}, & \text{for } \epsilon \ge \epsilon_n^*. \end{cases} \tag{C35}$$

Equation (C35) was obtained by noting that in (C34) the third term is less than the forth term evaluated at any odd k between $2^{n-1}+1$ and 2^n-1; and the first term is less than the third term if and only if both $k = i+1$ (since k is even) and (after some algebra) $\phi(\epsilon) < 0$ (i.e., $\epsilon < \epsilon_n^*$ via Lemma IV.1). If $i = 2^{n-1}-1$, then

$$\begin{aligned}\min_{k>i} \rho_n(i,k) &= \min \Bigg\{ \epsilon + 2^{-n-1}(i+k+1)\delta\Big|_{k=2^{n-1}}, \\ &\quad \min_{\substack{i<k \\ 2^{n-1}+1<k<2^n-1 \\ k \text{ odd}}} \epsilon + 2^{-n-1}(i+k+\delta)\delta - \frac{\epsilon(1-\epsilon)}{i-k} \Bigg\} \end{aligned} \tag{C36}$$

$$= \frac{1}{2}. \tag{C37}$$

Equation (C37) was obtained by noting that in (C36) the second term evaluated at any odd k between $2^{n-1}+1$ and 2^n-1 is always greater than the first term.

The ith encoding cell $R_n(i)$ is nonempty if and only if the conditions in (C20)–(C22) are each false. Suppose $0 \le \epsilon < \epsilon_n^*$. Then (C32), (C33), and (C35) imply (C20) is false for $i \neq 2^{n-1}-1$ if and only if

$$-\frac{2^n\epsilon^2}{\delta} < \delta + \frac{2^n\epsilon(2+\epsilon)}{1+2\epsilon} \tag{C38}$$

which is always true. If $i = 2^{n-1}-1$, then (C32) and (C37) imply (C20) is false if and only if

$$-\delta - \frac{2^n\epsilon^2}{\delta} < 2^{n-1}(1-\delta) = 2^n\epsilon \tag{C39}$$

which is always true. Equations (C35) and (C37) imply (C21) is always false since

$$\min_{k>i} \rho_n(i,k) > 0$$

by inspection. Equations (C32) and (C33) imply (C22) is false if and only if

$$i\delta^2 < 2^n(1-\epsilon)^2 \tag{C40}$$

which is always true. Hence, if $\epsilon \in [0, \epsilon_n^*)$, then $R_n(i)$ is nonempty.

Suppose $\epsilon_n^* \le \epsilon < 1/2$. Equations (C32) and (C35) imply (C20) is false (assuming $i \neq 1$ and $i \neq 2^{n-1}-1$) if and only if

$$\left((2i-2)\delta + \delta^2 + 2^{n+1}\epsilon\right)2^{-n-1} < \left((2i+2)\delta + \delta^2 + 2^{n+1}\epsilon\right)2^{-n-1} \tag{C41}$$

which is always true. If $i = 2^{n-1}-1$, then (C32) and (C37) imply (C20) is false if and only if

$$2^n - \delta(4-\delta) < 2^n \tag{C42}$$

which is always true. If $i = 1$, then (C33) and (C35) imply (C20) is false if and only if

$$-2^{n+1}\epsilon^2 < 2\delta^2 + \delta^3 + 2^{n+1}\epsilon\delta \tag{C43}$$

which is always true. Equations (C35) and (C37) imply (C21) is always false since

$$\min_{k>i} \rho_n(i,k) > 0$$

by inspection. Equation (C32) implies (C22) is false for $i \neq 1$ if and only if

$$(2i-2)\delta + \delta^2 < 2^{n+1}(1-\epsilon) \tag{C44}$$

which is always true. If $i = 1$, then (C40) implies that

$$\max_{k<i} \rho_n(1,k) < 1$$

and hence (C22) is false. Therefore, if $\epsilon \in [\epsilon_n^*, 1/2)$, then $R_n(i)$ is nonempty.

Case 3: i even , $2^{n-1} \le i \le 2^n-2$

Equations (5), (C18), and (C30) as well as Lemma III.5 imply (C45) at the bottom of the page. Equation (C45) implies (C46) at

$$\begin{aligned}\rho_n(i,k) &= \begin{cases} \epsilon + 2^{-n-1}(i+k+1+2\epsilon)\delta - \frac{\epsilon(1-\epsilon)}{i-k}, & \text{for } 0 \le k \le 2^{n-1}-2,\ k \text{ even} \\ \epsilon + 2^{-n-1}(i+k+1+2\epsilon)\delta, & \text{for } 2^{n-1} \le k \le 2^n-2,\ k \text{ even} \\ \epsilon + 2^{-n-1}(i+k+1)\delta, & \text{for } 1 \le k \le 2^{n-1}-1,\ k \text{ odd} \\ \epsilon + 2^{-n-1}(i+k+1)\delta - \frac{\epsilon(1-\epsilon)}{i-k+2\epsilon}, & \text{for } 2^{n-1}+1 \le k \le 2^n-1,\ k \text{ odd} \end{cases} \\ &= 1 - \rho_n(2^n-1-i, 2^n-1-k). \end{aligned} \tag{C45}$$

$$\max_{k<i}\rho_n(i,k) = 1-\min_{k<i}\rho_n(2^n-1-i,2^n-1-k) = \begin{cases} \left(i\delta+\frac{3\cdot 2^n\epsilon^2}{1+2\epsilon}\right)2^{-n}, & \text{for } i\neq 2^{n-1} \text{ and } \epsilon<\epsilon_n^* \\ \left((2i-2)\delta+1-4\epsilon^2+2^{n+1}\epsilon\right)2^{-n-1}, & \text{for } i\neq 2^{n-1} \text{ and } \epsilon\geq\epsilon_n^* \\ \frac{1}{2}, & \text{for } i=2^{n-1} \end{cases} \tag{C46}$$

$$\min_{k>i}\rho_n(i,k) = 1-\max_{k>i}\rho_n(2^n-1-i,2^n-1-k) = \begin{cases} \left((i+1)\delta+\frac{2^n\epsilon(2-3\epsilon)}{\delta}\right)2^{-n}, & \text{for } \epsilon<\epsilon_n^* \\ \left((2i+2)\delta+1-4\epsilon^2+2^{n+1}\epsilon\right)2^{-n-1}, & for\, i\neq 2^n-2 \text{ and } \epsilon\geq\epsilon_n^* \\ \left((2^n-1)\delta+\frac{2^n\epsilon(2-3\epsilon)}{\delta}\right)2^{-n}, & \text{for } i=2^n-2 \text{ and } \epsilon\geq\epsilon_n^* \end{cases} \tag{C47}$$

the top of the page, where (C46) follows from (C35) and (C37), as well as (C47) also at the top of the page, where (C47) follows from (C32) and (C33).

The ith encoding cell $R_n(i)$ is nonempty if and only if the conditions in (C20)–(C22) are each false. Equation (C45) implies (C20) is false if and only if

$$\min_{k<i}\rho_n(2^n-1-i,2^n-1-k) > \max_{k>i}\rho_n(2^n-1-i,2^n-1-k) \iff \min_{k>j}\rho_n(j,k) > \max_{k<j}\rho_n(j,k), \quad \text{for } 1\leq j\leq 2^{n-1}-1,\ j \text{ odd}$$

which is always true, as shown by (C38), (C39), (C41), (C42), and (C43). Equation (C45) implies (C21) is false if and only if

$$\max_{k>i}\rho_n(2^n-1-i,2^n-1-k)<1 \iff \max_{k<j}\rho_n(j,k)<1, \quad \text{for } 1\leq j\leq 2^{n-1}-1,\ j \text{ odd}$$

which is always true, as shown by (C40) and (C44). Equation (C45) implies (C22) is false if and only if

$$\min_{k<i}\rho_n(2^n-1-i,2^n-1-k)>0 \iff \min_{k>j}\rho_n(j,k)>0, \quad \text{for } 1\leq j\leq 2^{n-1}-1,\ j \text{ odd}$$

which is always true, as shown by inspection of (C35) and (C37). Hence, $R_n(i)$ is nonempty.

Case 4: i odd, $2^{n-1}+1\leq i\leq 2^n-1$

Equations (5), (C18), and (C23) as well as Lemma III.5 imply (C48) at the bottom of the page. Equation (C48) implies that

$$\max_{k<i}\rho_n(i,k)=1-\min_{k<i}\rho_n(2^n-1-i,2^n-1-k) \tag{C49}$$

$$= \left(i\delta+\frac{2^n\epsilon(2-3\epsilon)}{\delta}\right)2^{-n} \tag{C50}$$

where (C50) follows from (C25), and (assuming $i\neq 2^n-1$)

$$\min_{k>i}\rho_n(i,k)=1-\max_{k>i}\rho_n(2^n-1-i,2^n-1-k) \tag{C51}$$

$$= \left((i+1)\delta+\frac{3\cdot 2^n\epsilon^2}{1+2\epsilon}\right)2^{-n} \tag{C52}$$

where (C52) follows from (C24).

For $i\neq 2^n-1$, the ith encoding cell $R_n(i)$ is nonempty if and only if the conditions in (C20)–(C22) are each false. Equations (C49) and (C51) imply (C20) is false if and only if

$$\min_{k<i}\rho_n(2^n-1-i,2^n-1-k) > \max_{k>i}\rho_n(2^n-1-i,2^n-1-k) \iff \min_{k>j}\rho_n(j,k)>\max_{k<j}\rho_n(j,k), \quad \text{for } 2\leq j\leq 2^{n-1}-2,\ j \text{ even} \iff \epsilon<\epsilon_n^* \tag{C53}$$

where (C53) follows from (C26). Equation (C51) implies (C21) is false if and only if

$$\max_{k>i}\rho_n(2^n-1-i,2^n-1-k)<1 \iff \max_{k<j}\rho_n(j,k)<1, \quad \text{for } 2\leq j\leq 2^{n-1}-2,\ j \text{ even.} \tag{C54}$$

Equation (C28) implies (C54) is always true. Equation (C49) implies (C22) is false if and only if

$$\min_{k<i}\rho_n(2^n-1-i,2^n-1-k)>0 \iff \min_{k>j}\rho_n(j,k)>0, \quad \text{for } 2\leq j\leq 2^{n-1}-2,\ j \text{ even.} \tag{C55}$$

$$\rho_n(i,k) = \begin{cases} \epsilon+2^{-n-1}(i+k+1)\delta, & \text{for } 0\leq k\leq 2^{n-1}-2,\ k \text{ even} \\ \epsilon+2^{-n-1}(i+k+1)\delta+\frac{\epsilon(1-\epsilon)}{i-k-2\epsilon}, & \text{for } 2^{n-1}\leq k\leq 2^n-2,\ k \text{ even} \\ \epsilon+2^{-n-1}(i+k+\delta)\delta+\frac{\epsilon(1-\epsilon)}{i-k}, & \text{for } 1\leq k\leq 2^{n-1}-1,\ k \text{ odd} \\ \epsilon+2^{-n-1}(i+k+\delta)\delta, & \text{for } 2^{n-1}+1\leq k\leq 2^n-1,\ k \text{ odd} \end{cases}$$
$$=1-\rho_n(2^n-1-i,2^n-1-k). \tag{C48}$$

Equation (C27) implies that (C55) holds if and only if

$$\epsilon < \frac{1}{2^{(n/2)-(1/2)\log(2^n-1-i+1)}+2}. \tag{C56}$$

Lemma IV.1 implies that ϵ_n^* is smaller than the right-hand side of (C56) for $i \leq 2^n - 1$. Hence, if $\epsilon < \epsilon_n^*$, then (C56) holds and, therefore, $R_n(i)$ is nonempty for $i \neq 2^n - 1$ if and only if $\epsilon < \epsilon_n^*$.

For $i = 2^n - 1$, the conditions in (C20) and (C22) are equivalent and the condition in (C21) is always false. Therefore, the encoding cell $R_n(2^n - 1)$ is nonempty (from (C22) and (C49)) if and only if

$$\begin{aligned} &\min_{k<2^n-1} \rho_n(2^n-1-(2^n-1), 2^n-1-k) > 0 \\ &\iff \min_{k>0} \rho_n(0,k) > 0 \\ &\iff \epsilon < \frac{1}{2^{n/2}+2} \end{aligned} \tag{C57}$$

where (C57) follows from (C29). □

Proof of Theorem IV.4: The definition of r_c implies

$$r_c = \frac{1}{n}\sum_{i\in\Lambda} l(R_n(i)) \log \frac{1}{l(R_n(i))}. \tag{C58}$$

For $i \in \Lambda$, Theorem IV.2 and Corollary IV.3 give $l(R_n(i))$ as follows. If $n = 2$ and $\epsilon \in [0, 1/4)$, then

$$l(R_n(i)) = \begin{cases} \delta 2^{-n} - \frac{\epsilon^2}{\delta}, & \text{for } i = 0,3 \\ \frac{1}{2} - \delta 2^{-n} + \frac{\epsilon^2}{\delta}, & \text{for } i = 1,2. \end{cases}$$

If $n = 2$ and $\epsilon \in [1/4, 1/2)$, then

$$l(R_n(i)) = \frac{1}{2}, \quad \text{for } i = 1,2.$$

If $n \geq 3$ and $\epsilon \in [0, \epsilon_n^*)$, then

$$l(R_n(i)) = \begin{cases} \delta 2^{-n} - \frac{\epsilon^2}{\delta}, & \text{for } i = 0,\ 2^n-1 \\ \delta 2^{-n} + \frac{2\epsilon(1-\epsilon)}{1-4\epsilon^2}, & \text{for } 1 \leq i \leq 2^{n-1}-3,\ i \text{ odd; or} \\ & 2^{n-1}+2 \leq i \leq 2^n-2,\ i \text{ even} \\ \delta 2^{-n} - \frac{2\epsilon(1-\epsilon)}{1-4\epsilon^2}, & \text{for } 2 \leq i \leq 2^{n-1}-2,\ i \text{ even; or} \\ & 2^{n-1}+1 \leq i \leq 2^n-3,\ i \text{ odd} \\ \delta 2^{-n} + \frac{\epsilon^2}{\delta} + \epsilon, & \text{for } i = 2^{n-1}-1,\ 2^{n-1}. \end{cases}$$

If $n \geq 3$ and $\epsilon \in [\epsilon_n^*, 1/(2^{n/2}+2))$, then

$$l(R_n(i)) = \begin{cases} \delta 2^{-n} - \frac{\epsilon^2}{\delta}, & \text{for } i = 0,\ 2^n-1 \\ \delta 2^{-n} + 2^{-n-1}\delta^2 + \epsilon + \frac{\epsilon^2}{\delta}, & \text{for } i = 1,\ 2^n-2 \\ \delta 2^{1-n}, & \text{for } 3 \leq i \leq 2^{n-1}-3,\ i \text{ odd; or} \\ & 2^{n-1}+2 \leq i \leq 2^n-4,\ i \text{ even} \\ \delta 2^{1-n} - 2^{-n-1}\delta^2, & \text{for } i = 2^{n-1}-1,\ 2^{n-1}. \end{cases}$$

If $n \geq 3$ and $\epsilon \in [1/(2^{n/2}+2), 1/2)$, then

$$l(R_n(i)) = \begin{cases} \delta 2^{1-n} + 2^{-n-1}\delta^2 + \epsilon, & \text{for } i = 1,\ 2^n-2 \\ \delta 2^{1-n}, & \text{for } 3 \leq i \leq 2^{n-1}-3,\ i \text{ odd; or} \\ & 2^{n-1}+2 \leq i \leq 2^n-4,\ i \text{ even} \\ \delta 2^{1-n} - 2^{-n-1}\delta^2, & \text{for } i = 2^{n-1}-1,\ 2^{n-1}. \end{cases}$$

The result follows from (C58) and routine algebra. □

APPENDIX IV
LEMMAS AND PROOFS FOR SECTION V

Lemma IV.1: $z_n(i) \cdot \alpha_n(i, I_r(i)) = \beta_n(i, I_r(i))$.

Proof of Lemma IV.1: Let i and j denote the indices of two adjacent, nonempty encoding cells. Then for all $x \in \overline{R}_n(i)$, the weighted nearest neighbor condition implies that

$$\alpha_n(i,j)x \geq \beta_n(i,j).$$

Assume, without loss of generality, that $\alpha_n(i,j) < 0$. Then

$$x \leq \frac{\beta_n(i,j)}{\alpha_n(i,j)}$$

for all $x \in \overline{R}_n(i)$. The weighted nearest neighbor condition also implies that

$$\alpha_n(j,i)x \geq \beta_n(j,i)$$

for all $x \in \overline{R}_n(j)$, or equivalently that

$$x \geq \frac{\beta_n(j,i)}{\alpha_n(j,i)}$$

for all $x \in \overline{R}_n(j)$ because

$$\alpha_n(j,i) = -\alpha_n(i,j) > 0.$$

Note, however, that

$$\frac{\beta_n(i,j)}{\alpha_n(i,j)} = \frac{\beta_n(j,i)}{\alpha_n(j,i)}.$$

Hence,

$$\frac{\beta_n(i,j)}{\alpha_n(i,j)}$$

must be the boundary between $R_n(i)$ and $R_n(j)$, for otherwise they cannot be adjacent. The lemma now follows from the definition of $z_n(i)$. □

Proof of Lemma V.1: From (1), we have

$$D_{EO}^{(\pi_n)} = \sum_{i\in\Lambda}\sum_{j=0}^{2^n-1} p_n(\pi_n(j)|\pi_n(i)) \int_{R_n(i)} (x - y_n(j))^2\, dx. \tag{D1}$$

Substituting

$$y_n(j) = (j + (1/2))2^{-n}$$

into (D1), expanding the squared term, integrating and then summing over constant terms, and expressing the result in terms of $z_n(i)$ and $I_l(i)$ gives

$$\begin{aligned} D_{EO}^{(\pi_n)} = &\frac{1}{3} - 2^{-n-1} + 2^{-2n-2} \\ &- 2^{-n}\sum_{i\in\Lambda}[z_n^2(i) - z_n^2(I_l(i))]\sum_{j=0}^{2^n-1} j p_n(\pi_n(j)|\pi_n(i)) \\ &+ 2^{-2n}\sum_{i\in\Lambda}[z_n(i) - z_n(I_l(i))]\sum_{j=0}^{2^n-1}(j+j^2)p_n(\pi(j)|\pi(i)) \end{aligned} \tag{D2}$$

where (D2) follows since each $R_n(i)$ is an interval. Re-expressing the elements of (D2) which include $I_l(i)$ in terms of $I_r(i)$, collecting terms using the definitions of $\alpha_n(i,k)$

and $\beta_n(i,k)$ in (4) and (5), respectively, and simplifying with Lemma IV.1 gives

$$\begin{aligned} D_{EO}^{(\pi_n)} &= \frac{1}{3} - 2^{-n-1} + 2^{-2n-2} \\ &\quad + 2^{-n}\left[\sum_{i\in V} z_n^2(i)\cdot\alpha_n(i,I_r(i)) - \sum_{j=0}^{2^n-1} j p_n(\pi_n(j)|\pi_n(I_1))\right] \\ &\quad + 2^{-2n}\sum_{j=0}^{2^n-1}(j+j^2)p_n(\pi_n(j)|\pi_n(I_1)). \qquad \square \end{aligned}$$

Proof of Theorem V.4: Let

$$\hat{p}_n(j|i) = p_n(\pi_n^{(\text{CNC})}(j)|\pi_n^{(\text{CNC})}(i)).$$

Case 1: $0 \le \epsilon < \epsilon_n^*$

Theorem IV.2 and Corollary IV.3 imply that

$$\begin{aligned} V &= \{1,2,\ldots,2^n-2\} \\ I_r(i) &= i+1 \\ I_1 &= 2^n-1. \end{aligned}$$

Hence, using Lemmas III.3 and III.5 to evaluate the last two sums in Lemma V.1 and (C18) to simplify the first sum in Lemma V.1 gives

$$\begin{aligned} D_{EO}^{(\text{CNC})} &= \frac{1}{3} - 2^{-n-1} + 2^{-2n-2} \\ &\quad + (2^{-2n}-2^{-n})\left[(2^n-1)\epsilon + \delta(2^n-1-\epsilon)\right] \\ &\quad + 2^{-2n}\Bigg(\epsilon\left[\left(\frac{2^{2n}-1}{3}\right)+2^n+1\right] \\ &\quad + \frac{\epsilon^2}{3}(2^{2n+1}-21\cdot 2^n-2) + 3\cdot 2^{n+1}\epsilon^3 \\ &\quad + (2^n-1)\delta[2\epsilon(2^n-1)-2\epsilon\delta] + (2^n-1)^2\delta^2\Bigg) \\ &\quad - 2^{-n}\left(\sum_{\substack{i\in V\\ i \text{ even}}} z_n^2(i)\cdot\delta^2 + \sum_{\substack{i\in V\\ i \text{ odd}}} z_n^2(i)\cdot\delta(1+2\epsilon)\right) \\ &= \frac{2^{-2n}}{3(1+2\epsilon)}\big((1/4)+(2^{2n}+(5/2))\epsilon \\ &\quad - (2^{2n+1}-15\cdot 2^n+4)\epsilon^2 \\ &\quad + 6(2^{2n}-2^{n+2}-4)\epsilon^3 + (2^n-4)(2^n-2)\epsilon^4 \\ &\quad -12(2^n-4)\epsilon^5\big) \qquad \text{(D3)} \\ &= D_1(n,\epsilon) \end{aligned}$$

where (D3) follows from considerable arithmetic and using (from Theorem IV.2)

$$z_n(i) = \begin{cases} \left((i+1)\delta - \frac{2^n\epsilon^2}{\delta}\right)2^{-n}, & \text{for } 0 \le i \le 2^{n-1}-2,\ i \text{ even} \\ \left((i+1)\delta + \frac{2^n\epsilon(2-3\epsilon)}{\delta}\right)2^{-n}, & \text{for } 2^{n-1} \le i \le 2^n-2,\ i \text{ even} \\ \left((i+1)\delta + \frac{2^n\epsilon(2+\epsilon)}{1+2\epsilon}\right)2^{-n}, & \text{for } 1 \le i \le 2^{n-1}-3,\ i \text{ odd} \\ \frac{1}{2}, & \text{for } i = 2^{n-1}-1 \\ \left((i+1)\delta + \frac{3\cdot 2^n\epsilon^2}{1+2\epsilon}\right)2^{-n}, & \text{for } 2^{n-1}+1 \le i \le 2^n-3,\ i \text{ odd.} \end{cases}$$

Case 2: $\frac{1}{2^{n/2}+2} \le \epsilon < 1/2$

Theorem IV.2 and Corollary IV.3 imply that

$$\begin{aligned} V &= \{1,3,5,\ldots,2^{n-1}-1\} \\ &\quad \cup \{2^{n-1},2^{n-1}+2,2^{n-1}+4,\ldots,2^n-4\} \\ I_r(i) &= i+2, \quad \text{for } i \in \{i : i\in V, i\ne 2^{n-1}-1\} \\ I_r(2^{n-1}-1) &= 2^{n-1} \\ I_1 &= 2^n-2. \end{aligned}$$

Hence, using Lemmas III.3 and III.5 to evaluate the last two sums in Lemma V.1; using (C18) and Theorem IV.2 to simplify the first sum in Lemma V.1; and collecting terms according to which power of ϵ they contain gives

$$\begin{aligned} D_{EO}^{(\text{CNC})} &= \frac{1}{3} - 3\cdot 2^{-n-1} + 9\cdot 2^{-2n-2} \\ &\quad + \left(-\frac{2}{3} + 6\cdot 2^{-n} - 34\cdot\frac{2^{-2n}}{3}\right)\epsilon \\ &\quad + \left(\frac{2}{3} - 7\cdot 2^{-n} + 52\cdot\frac{2^{-2n}}{3}\right)\epsilon^2 \\ &\quad + (2^{-n+1}-2^{-2n+3})\epsilon^3 \\ &\quad - 2^{-n}\left[\frac{1-4\epsilon^2}{4} + 2\delta\sum_{\substack{i\in V\\ i\ne 2^{n-1}-1}} z_n^2(i)\right]. \qquad \text{(D4)} \end{aligned}$$

Theorem IV.2 shows that

$$z_n(i) = 1 - z_n(2^n-3-i)$$

for $2^{n-1} \le i \le 2^n-4$ and i even when

$$\frac{1}{2^{n/2}+2} \le \epsilon < 1/2.$$

Therefore, using Theorem IV.2 to evaluate $z_n(i)$, the last term in (D4) can be rewritten as

$$\begin{aligned} &-2^{-n}\left[\frac{1-4\epsilon^2}{4} + 2\delta\left(\sum_{\substack{i=1\\ i \text{ odd}}}^{2^{n-1}-3} z_n^2(i) + \sum_{\substack{i=2^{n-1}\\ i \text{ even}}}^{2^n-4} z_n^2(i)\right)\right] \\ &= \left(-\frac{1}{3} + 3\cdot 2^{-n-1} - \frac{23}{3}\cdot 2^{-2n-2} + 2^{-3n}\right) \\ &\quad + \left(1 - 7\cdot 2^{-n} + 29\cdot 2^{-2n-1} - 5\cdot 2^{-3n+1}\right)\epsilon \\ &\quad + \left(-1 + 13\cdot 2^{-n} - 21\cdot 2^{-2n+1} + 5\cdot 2^{-3n+3}\right)\epsilon^2 \\ &\quad + \left(\frac{2}{3} - 3\cdot 2^{-n+2} + \frac{43}{3}\cdot 2^{-2n+2} - 5\cdot 2^{-3n+4}\right)\epsilon^3 \\ &\quad + (2^{-n+2} - 9\cdot 2^{-2n+2} + 5\cdot 2^{-3n+4})\epsilon^4 \\ &\quad + (2^{-2n+3} - 2^{-3n+5})\epsilon^5 \qquad \text{(D5)} \end{aligned}$$

where (D5) follows after considerable arithmetic. Substituting (D5) for the last term in (D4) and collecting terms gives

$$\begin{aligned} D_{EO}^{(\text{CNC})} &= \frac{2^{-3n}}{3}\Big(2^n + 3 + [(2^n-3)(2^{2n}+10) - 2^{n-1}]\epsilon \\ &\quad - [(2^n-6)(2^n-5)(2^n-4)\delta - 3\cdot 2^{2n}]\epsilon^2 \\ &\quad + 12(2^n-5)(2^n-4)\epsilon^4 + 24(2^n-4)\epsilon^5\Big) \qquad \text{(D6)} \\ &= D_3(n,\epsilon). \end{aligned}$$

Case 3: $\epsilon_n^* \leq \epsilon < \frac{1}{2^{n/2}+2}$
Theorem IV.2 and Corollary IV.3 imply that

$$\begin{aligned} V &= \{0, 1, 3, 5, \ldots, 2^{n-1} - 1\} \\ &\quad \cup \{2^{n-1}, 2^{n-1} + 2, 2^{n-1} + 4, \ldots, 2^n - 2\} \\ I_r(0) &= 1 \\ I_r(2^n - 2) &= 2^n - 1 \\ I_1 &= 2^n - 1. \end{aligned}$$

Theorem IV.2 also shows that if $i \in V - \{0, 2^n - 2\}$, then the expressions for $z_n(i)$ and $I_r(i)$ are the same as the expressions for $z_n(i)$ and $I_r(i)$ in Case 2. Hence, Lemma V.1 gives

$$\begin{aligned} D_{EO}^{(\mathrm{CNC})} &= D_3(n, \epsilon) + 2^{-n}\left[z_n^2(0)\alpha_n\left(0, I_r(0)\right)\right. \\ &\quad \left. + z_n^2(2^n - 2)\alpha_n\left(2^n - 2, I_r(2^n - 2)\right)\right] \\ &\quad + (2^{-2n} - 2^{-n}) \sum_{j=0}^{2^n-1} j\left[\hat{p}_n(j|2^n - 1) - \hat{p}_n(j|2^n - 2)\right] \\ &\quad + 2^{-2n} \sum_{j=0}^{2^n-1} j^2\left[\hat{p}_n(j|2^n - 1) - \hat{p}_n(j|2^n - 2)\right]. \quad \text{(D7)} \end{aligned}$$

Simplifying (D7) with (4) and Theorem IV.2, using (C18) to evaluate α_n, and using Lemma III.5 to calculate the sum over j^2 gives

$$\begin{aligned} D_{EO}^{(\mathrm{CNC})} &= D_3(n, \epsilon) - 2^{-n}\left[z_n^2(0)\delta^2 + (1 - z_n(0))^2\delta^2\right] \\ &\quad + (2^{-2n} - 2^{-n})\delta^2 \\ &\quad + 2^{-2n}\left[2^{n+1}\epsilon + \frac{\epsilon^2}{3}(-30 \cdot 2^n + 12)\right. \\ &\quad + \epsilon^3(6 \cdot 2^{n+1} - 8) \\ &\quad \left. + \delta[2\epsilon(2^n - 1) - 2(2^{n+1} - 3)\epsilon\delta] + (2^{n+1} - 3)\delta^2\right]. \quad \text{(D8)} \end{aligned}$$

Theorem IV.2 implies

$$\begin{aligned} z_n(0) = 1 - \frac{2^{-n}}{\delta^2}&\left(2^n - 1 + (-2^{n+2} + 6)\epsilon\right. \\ &\left. + (5 \cdot 2^n - 12)\epsilon^2 + (-2^{n+1} + 8)\epsilon^3\right). \quad \text{(D9)} \end{aligned}$$

Substituting (D6) and (D9) into (D8) and performing considerable arithmetic gives

$$D_{EO}^{(CNC)} = D_2(n, \epsilon). \qquad \square$$

Proof of Lemma V.5: Let $N = 2^n$. The proof is straightforward for the case $N = 4$, so assume $N \geq 8$. Note that

$$\begin{aligned} g_n(N^{-2}) &= -4N^{-1} - 2N^{-2} + 8N^{-3} + 9N^{-4} \\ &\quad - 2N^{-5} + 4N^{-7} - 16N^{-8} \\ g_n(2N^{-2}) &= 1 - 8N^{-1} - 8N^{-2} + 32N^{-3} + 40N^{-4} \\ &\quad - 16N^{-5} + 64N^{-7} - 256N^{-8}. \end{aligned}$$

We have $g_n(N^{-2}) < 0$ since

$$-4N^{-1} + 8N^{-3} + 9N^{-4} < 0$$

and

$$-2N^{-5} + 4N^{-7} < 0$$

and we have $g_n(2N^{-2}) > 0$ for $N > 8$ since

$$\begin{aligned} 64N^{-7} - 256N^{-8} &> 0, \\ 40N^{-4} - 16N^{-5} &> 0, \end{aligned}$$

and

$$1 - 8N^{-1} - 8N^{-2} > 0.$$

Thus, the function g_n has a root in

$$(N^{-2}, 2N^{-2}) \subset (0, 1/2)$$

for $N > 8$, and it has a root in $(0, 1/2)$ for $N = 8$ since $g_3(0) = -1 < 0$ and $g_3(1/2) = 8$. The first three derivatives of g_n are

$$\begin{aligned} g_n'(\epsilon) &= 16(N - 4)\epsilon^3 + 3N(N - 2)\epsilon^2 \\ &\quad - 4(N^2 - 4N - 4)\epsilon + N(N - 4) \\ g_n''(\epsilon) &= 48(N - 4)\epsilon^2 + 6N(N - 2)\epsilon \\ &\quad - 4(N^2 - 4N - 4) \\ g_n'''(\epsilon) &= 96(N - 4)\epsilon + 6N(N - 2). \end{aligned}$$

Since $g_n''' > 0$ on $[0, 1/2]$ and

$$g_n''(1/2) = -5N^2 + 38N - 16 < 0$$

we must have $g_n'' < 0$ on $[0, 1/2]$, which implies $g_n' = 0$ at most once on $[0, 1/2]$. Therefore, since

$$g_n(0) = -1 < 0$$

and

$$g_n(1/2) = N^2/8 > 0$$

the function g_n has exactly one root on $[0, 1/2)$, which implies that $g_n(\epsilon) < 0$ on if and only if $\epsilon < \hat{\epsilon}_n$. $\square$

Note that the root $\hat{\epsilon}_n$ of g_n could be found explicitly using the formula for the general solution to a quartic polynomial equation.

Proof of Theorem V.6: If $n = 2$, then Theorem V.4 implies that the value of $D_{EO}^{(CNC)}$ is the same for

$$0 \leq \epsilon < \epsilon_n^*$$

and

$$\epsilon_n^* \leq \epsilon < \frac{1}{2^{n/2} + 2}$$

which gives

$$D_{EO}^{(\mathrm{CNC})} = \begin{cases} \frac{1+72\epsilon-48\epsilon^2}{192}, & \text{for } 0 \leq \epsilon < \frac{1}{4} \\ \frac{7+24\epsilon+48\epsilon^2}{192}, & \text{for } \frac{1}{4} \leq \epsilon < \frac{1}{2}. \end{cases}$$

Theorem V.3 gives

$$D_{EO}^{(\mathrm{NBC})} = \frac{1+60\epsilon-24\epsilon^2+48\epsilon^3}{192}.$$

Therefore, for $n=2$

$$\begin{aligned} &D_{EO}^{(\mathrm{CNC})} - D_{EO}^{(\mathrm{NBC})} \\ &\quad = \begin{cases} \frac{\epsilon(1-2\epsilon-4\epsilon^2)}{16}, & \text{for } 0 \le \epsilon < \frac{1}{4} \\ \frac{(1-2\epsilon)^3}{32}, & \text{for } \frac{1}{4} \le \epsilon < \frac{1}{2} \end{cases} \\ &\quad > 0. \end{aligned}$$

Now let $n \ge 3$.

Case 1: $0 \le \epsilon < \epsilon_n^*$

Theorems V.3 and V.4 imply that

$$\begin{aligned} &D_{EO}^{(\mathrm{NBC})} - D_{EO}^{(\mathrm{CNC})} \\ &\quad = \frac{2^{-2n}\epsilon}{1+2\epsilon}\Big[-1+(2^{2n}-2^{n+2})\epsilon \\ &\qquad -(2^{2n+1}-2^{n+3}-8)\epsilon^2+(2^{2n}-2^{n+1})\epsilon^3 \\ &\qquad +(2^{n+2}-16)\epsilon^4\Big] \end{aligned}$$

which (by Lemma V.5) is positive if and only if $\epsilon > \hat{\epsilon}_n$.

Case 2: $\epsilon_n^* \le \epsilon < \frac{1}{2^{n/2}+2}$

Theorems V.3 and V.4 imply that

$$\begin{aligned} \frac{D_{EO}^{(NBC)} - D_{EO}^{(CNC)}}{2^{-n}\delta^2} &= (1-\epsilon)\epsilon + \frac{2\epsilon^4}{\delta^2} \\ &+ 2^{-n-2}\left[\delta^3 - 2(1+6\epsilon)\right] + 2^{-2n}(1+2\epsilon)\delta^2. \end{aligned} \tag{D10}$$

For $n=3$, the right-hand side of (D10) is

$$\begin{aligned} &\frac{-1+26\epsilon-44\epsilon^2-8\epsilon^3}{64} + \frac{2\epsilon^4}{\delta^2} \\ &\quad = 2\epsilon\left(\frac{\epsilon^3}{(1-2\epsilon)^2} + \frac{1-4\epsilon}{16}\right) + \frac{\phi_3(\epsilon)}{64} > 0 \end{aligned} \tag{D11}$$

where (D11) follows from $\phi_3(\epsilon) \ge 0$ and

$$\epsilon < 1/(2+\sqrt{8}) < 1/4.$$

For $n \ge 4$, the right-hand side of (D10) can be lower-bounded as

$$\begin{aligned} &\epsilon(1-\epsilon) + \frac{2\epsilon^4}{\delta^2} + 2^{-n-2}\left[\delta^3 - 2(1+6\epsilon)\right] \\ &\quad + 2^{-2n}(1+2\epsilon)\delta^2 \\ &\ge \epsilon - \epsilon^2 + 2\epsilon^4 + 2^{-n-2}\left(-1-18\epsilon+12\epsilon^2-8\epsilon^3\right) \\ &\quad + 2^{-2n}\left(1-2\epsilon-4\epsilon^2+8\epsilon^3\right) && \text{(D12)} \\ &\ge \epsilon 2^{-n-1}\left[\frac{\phi_n(\epsilon)}{2\epsilon} + 2^n(1-\epsilon) - 10 \right. \\ &\qquad \left. -(1+2\epsilon)2^{-n+2} - 2\epsilon\right] && \text{(D13)} \\ &\ge \epsilon 2^{-n-1}\left[2^4(1-(1/6)) - 10 \right. \\ &\qquad \left. -(1+2(1/6))2^{-2} - 2(1/6)\right] && \text{(D14)} \\ &> 0 \end{aligned}$$

where (D12) follows from $\frac{2\epsilon^4}{\delta^2} \ge 2\epsilon^4$ and simplifying; (D13) follows by eliminating all positive terms except ϵ, and then simplifying; and (D14) follows from the fact that $\phi_n(\epsilon) \ge 0$ when $\epsilon \ge \epsilon_n^*$ (by Lemma IV.1), and the fact that

$$\epsilon < \frac{1}{2^{n/2}+2} \le \frac{1}{2^{4/2}+2} = 1/6$$

for all $n \ge 4$.

Case 3: $\frac{1}{2^{n/2}+2} \le \epsilon < 1/2$

Theorems V.3 and V.4 imply that

$$\begin{aligned} &D_{EO}^{(\mathrm{NBC})} - D_{EO}^{(\mathrm{CNC})} \\ &\quad = 2^{-n}\delta^2\Big[\epsilon(1-\epsilon) - 2^{-n-2}(1+18\epsilon-28\epsilon^2+8\epsilon^3) \\ &\qquad\qquad - 2^{-2n}\delta^3\Big] \\ &\quad > 2^{-n}\delta^2\Big[\epsilon(1-\epsilon) - 2^{-n-2}\cdot 2^{2.1} - 2^{-2n}\Big] && \text{(D15)} \\ &\quad > 0 && \text{(D16)} \end{aligned}$$

where (D15) follows from the fact that $\delta^3 < 1$ and

$$\log(1+18\epsilon-28\epsilon^2+8\epsilon^3) < 2.1$$

for $0 \le \epsilon \le 1/2$; and (D16) follows from the facts that $\epsilon(1-\epsilon)$ is monotone increasing with ϵ and

$$\epsilon(1-\epsilon) - 2^{-n+0.1} - 2^{-2n} > 0$$

for

$$\epsilon = \frac{1}{2^{n/2}+2}$$

and $n \ge 3$. □

ACKNOWLEDGMENT

The authors would like to thank two anonymous reviewers for their excellent suggestions and careful reading of the manuscript.

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