Quantizers With Uniform Encoders and Channel Optimized Decoders

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***Abstract*—Scalar quantizers with uniform encoders and channel optimized decoders are studied for uniform sources and binary symmetric channels. It is shown that the natural binary code (NBC) and folded binary code (FBC) induce point density functions that are uniform on proper subintervals of the source support, whereas the Gray code (GC) does not induce a point density function. The mean-squared errors (MSEs) for the NBC, FBC, GC, and for randomly chosen index assignments are calculated and the NBC is shown to be mean-squared optimal among all possible index assignments, for all bit-error rates and all quantizer transmission rates. In contrast, it is shown that almost all index assignments perform poorly and have degenerate codebooks.**

***Index Terms*—Point density function, quantization asymptotics, source channel coding.**

I. Introduction

THE most basic source and quantizer are the uniform scalar source and the uniform scalar quantizer. If the source is uniform on $[0,1]$, for example, then an n-bit uniform quantizer has equally spaced encoding cells of size 2^{-n} and has equally spaced output points which are the centers of the encoding cells. For this source, the mean-squared distortion of this quantizer is known exactly when there is no channel noise, and is known to be minimal among all quantizers.

In the presence of channel noise, one approach to improving system performance is to add explicit error control coding, so that some of the transmission rate is devoted toward source coding and some toward channel coding. Drawbacks of this include the added complexity and delay of channel decoding.

An alternative low-complexity approach in the presence of channel noise is to add to the quantizer an index assignment, which permutes the binary words associated with each encoding cell prior to transmission over the channel, and then unpermutes the binary words at the receiver prior to assigning a reproduction point at the output. The cells are assumed to be labeled in increasing order from left to right, before the index assignment. Examples of index assignments include the natural binary code (NBC), the folded binary code (FBC), and the Gray code (GC). The benefit of an index assignment is derived from the fact that reproduction codepoints that are relatively close on the real line can be assigned binary words which are close in the Hamming sense (i.e., in the number of same bits) on average. Thus, when channel errors occur, the mean-squared error (MSE) impact on the quantizer is reduced.

Yamaguchi and Huang [8] and Huang [9] derived formulas for the MSE of uniform scalar quantizers and uniform sources for the NBC, the GC, and for a randomly chosen index assignment on a binary symmetric channel. They also asserted (without a published proof) the optimality of the NBC for the binary symmetric channel. Crimmins *et al.* [1] studied the uniform scalar quantizer for the uniform source and proved the Yamaguchi–Huang assertion, that the NBC is the best possible index assignment in the mean-squared sense for the binary symmetric channel, for all bit-error probabilities, and all quantizer rates. McLaughlin, Neuhoff, and Ashley [3] generalized this result for certain uniform vector quantizers and uniform vector sources. Other than these papers, there are no others presently known in the literature giving index assignment optimality results.

There have been some analytic studies on the performance of various index assignments. Hagen and Hedelin [7] used Hadamard transforms to study certain lattice-type quantizers with index assignments on noisy channels. Knagenhjelm and Agrell [10] introduced an analytic method of approximating the quality of an index assignment using Hadamard transforms. Skoglund [12] provided index assignment analysis for more general channels and sources. In [4], explicit MSE formulas were computed for uniform sources on binary asymmetric channels with various structured classes of index assignments. In [5], it was shown that for the uniform source and uniform quantizer the MSE resulting from a randomly chosen index assignment was, on average, equal in the limit of large n to that of the worst possible index assignment. In this sense, the result showed that randomly chosen index assignments are asymptotically bad. A number of papers have also studied algorithmic techniques for designing good index assignments for particular sources and channels (see the citations in [6, p. 2372]).

While index assignments can improve the robustness of quantizers designed for noiseless channels to the degradation caused by channel noise, another low-complexity approach is to use quantizers whose encoders and/or decoders are designed for the channel's statistical behavior. It is known that an optimal quantizer for a noiseless channel must satisfy what are known as "nearest neighbor" and "centroid" conditions on its encoder and decoder, respectively [2]. For discrete memoryless channels, it is known that an optimal quantizer must satisfy what we

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call "weighted nearest neighbor" and "weighted centroid" conditions on its encoder and decoder, respectively (see [11] for example). Even for uniform scalar sources, the resulting quantizers in general do not have uniform encoding cells nor equally spaced reproduction codepoints. In fact, very little is presently understood analytically about quantizers for noisy channels beyond the NBC optimality results previously mentioned for uniform quantizers.

In the present paper, we attempt to move a step closer toward understanding optimal quantization for noisy channels by examining the structure of quantizers with uniform encoders and channel optimized decoders (i.e., that satisfy the weighted centroid condition), for uniform sources on $[0,1]$ and for certain previously studied index assignments. In particular, we study the high-resolution distribution of codepoints for such quantizers and the resulting distortions. Slightly more general, but notationally cumbersome results could also be easily obtained from our results by allowing the source to be confined to any bounded interval instead of just $[0,1]$.

An important tool in analyzing the performance of quantizers is the concept of point density functions. Point density functions characterize the high-resolution distribution of scalar quantizer codepoints. As a result, they provide insight about the asymptotic behavior of scalar quantizer codebooks and encoding cells. Point density functions also are useful in analyzing the distortion of quantizers. For example, Bennett's integral [2, p. 163] gives the average distortion in the high-resolution case for a nonuniform quantizer in terms of a point density function, source distribution, and size of the quantizer codebook (see [6] for more details). For uniform quantizers, the computation of a point density function is trivial. For nonuniform quantizers, however, point density functions are not always guaranteed to exist, and when they do, their computation can be difficult.

Point density functions depend on the quantizer decoders. Channel optimized quantizer decoders, in turn, depend on the source, the quantizer encoder, the channel, and the index assignment. For this paper, we assume a uniform source on $[0,1]$, a uniform quantizer encoder, a channel optimized quantizer decoder, and a binary symmetric channel with bit-error probability ϵ. An index assignment maps source codewords to channel codewords. The quantizer has 2^n encoding cells, and index assignments are one-to-one maps from the index of an encoding cell to a binary word of length n. These words are transmitted across the channel and decoded according to the weighted centroid condition.

Certain results we obtain are somewhat counter-intuitive. For example, we show that for a binary symmetric channel with bit-error probability ϵ, quantizers using the NBC index assignment and n bits of resolution have codepoints uniformly distributed on the interval $[\epsilon+\delta, 1-\epsilon-\delta]$ where

$$\delta = \frac{1-2\epsilon}{2^{n+1}}.$$

This is peculiar in light of the fact that the source is uniformly distributed on the interval $[0,1]$, and yet asymptotically as $n \to \infty$ no codepoints are located within a distance of ϵ from 0 or 1. The lack of codepoints in regions of positive source probability is due to the reduction in average distortion that results from moving codepoints closer to the source mean (by the weighted centroid condition), to avoid large jumps in Euclidean distance from channel errors. The weighted centroid condition dictates this movement of codepoints to minimize average distortion for a given quantizer encoder. A similar result occurs for the FBC.

For the GC index assignment, we show that, in fact, no point density function exists. In other words, the location of codepoints cannot be described according to a point density function as $n \to \infty$. The structure of the GC simply does not allow the histogram of codepoint locations to converge to a smooth function in the limit of high resolution.

We also show that asymptotically, almost all index assignments give rise to quantizers which have almost all of their codepoints clustered very close to the source's mean value (i.e., $1/2$). Thus, almost all index assignments are bad. As n grows, the clustering of codepoints becomes tighter and tighter. This contrasts with the NBC and the FBC cases where the codepoints remain uniformly distributed on proper subsets of $[0,1]$ no matter how large n becomes. An additional curiosity we show is that among all possible index assignments, the NBC is optimal despite its lack of codepoints within ϵ of 0 or 1.

Our main results for quantizers with uniform encoders and channel optimized decoders are the following. First, we show that the NBC index assignment yields a uniform point density function on the interval $(\epsilon, 1-\epsilon)$ (Theorem III.3), the FBC index assignment yields a uniform point density function on a union of two proper subintervals of $[0,1]$ (Theorem IV.2), the GC index assignment does not yield a point density function (Theorem V.10), and an arbitrarily large fraction of all index assignments have an arbitrarily large fraction of codepoints arbitrarily close to the source mean as $n \to \infty$ (Theorem VI.4). Then we extend a result in [5] by showing that most index assignments are asymptotically bad (Theorem VII.2), and we extend results in [4], [8], and [9] by computing the MSE resulting from the NBC (Theorem VII.4), the FBC (Theorem VII.6), the GC (Theorem VII.8), and a randomly chosen index assignment (Theorem VII.10). As comparisons, we state previously known MSE formulas for channel unoptimized decoders (i.e., that satisfy the centroid condition), for the NBC (Theorem VII.3), the FBC (Theorem VII.5), the GC (Theorem VII.7), and for a randomly chosen index assignment (Theorem VII.9). Finally, we extend the (uniform scalar quantizer) proof in [3] by showing that the NBC is an optimal index assignment for all bit-error rates and all quantizer transmission rates (Theorem VII.12).

The paper is organized as follows. Section II gives definitions and notation. Section III gives NBC results, Section IV gives FBC results, Section V gives GC results, Section VI considers arbitrarily selected index assignments, and Section VII gives distortion analysis.

II. Preliminaries

A *rate n quantizer* on $[0,1]$ is a mapping

$$\mathcal{Q}: [0,1] \longrightarrow \{y_n(0), y_n(1), \ldots, y_n(2^n-1)\}.$$

The real-valued quantities $y_n(i)$ are called *codepoints* and the set

$$\{y_n(0), \ldots, y_n(2^n-1)\}$$

is called a *codebook*. For a noiseless channel, the quantizer $\mathcal{Q}$ is the composition of a *quantizer encoder* and a *quantizer decoder*. These are, respectively, mappings

$$\mathcal{Q}_e: [0,1] \to \{0,1,\ldots,2^n-1\}$$
$$\mathcal{Q}_d: \{0,1,\ldots,2^n-1\} \to \{y_n(0), y_n(1), \ldots, y_n(2^n-1)\}$$

such that

$$\mathcal{Q}_d(i) = y_n(i)$$

for all i. For each i the set

$$\mathcal{Q}^{-1}(y_n(i)) = \mathcal{Q}_e^{-1}(\mathcal{Q}_d^{-1}(y_n(i)))$$

is called the ith encoding *cell*. The quantizer encoder is said to be *uniform* if for each i

$$\mathcal{Q}^{-1}(y_n(i)) \supseteq (i2^{-n}, (i+1)2^{-n}).$$

The *nearest neighbor* cells of a rate n quantizer are the sets

$$R_n(i) = \{x: |y_n(i) - x| < |y_n(j) - x|, \quad \forall\, j \neq i\}$$

for $0 \le i \le 2^n - 1$. Let m denote Lebesgue measure and for each i let

$$\mu_n(i) = m(R_n(i)).$$

A quantizer's encoder is said to satisfy the *nearest neighbor condition* if for each i

$$\mathcal{Q}^{-1}(y_n(i)) \supseteq R_n(i).$$

That is, its encoding cells are essentially nearest neighbor cells (boundary points can be assigned arbitrarily).

For a given n, i, and source random variable X, the *centroid* of the ith cell of the quantizer $\mathcal{Q}$ is the conditional mean

$$c_n(i) = E\left[X \mid \mathcal{Q}(X) = y_n(i)\right].$$

The quantizer decoder is said to satisfy the *centroid condition* if the codepoints satisfy

$$y_n(i) = c_n(i)$$

for all i. A quantizer is *uniform* if the encoder is uniform and for each i the decoder codepoint $y_n(i)$ is the midpoint of the cell $\mathcal{Q}^{-1}(y_n(i))$. It is known that if a quantizer minimizes the MSE for a given source and a noiseless channel, then it satisfies the nearest neighbor and centroid conditions [2]. In particular, if the source is uniform, then a uniform quantizer satisfies the nearest neighbor and centroid conditions.

For a rate-n quantizer, an *index assignment* π_n is a permutation of the set $\{0,1,\ldots,2^n-1\}$. Let S_{2^n} denote the set of all $2^n!$ such permutations. For a noisy channel, a random variable $X \in [0,1]$ is quantized by transmitting the index

$$I = \pi_n\left(\mathcal{Q}_e(X)\right)$$

across the channel, receiving index J from the channel, and then decoding the codepoint

$$y_n(\pi_n^{-1}(J)) = \mathcal{Q}_d(\pi_n^{-1}(J)).$$

We impose the following monotonicity constraint on quantizer encoders in order to be able to unambiguously refer to certain index assignments: For all $s, t \in [0,1]$

$$\text{if } s < t, \text{then } \mathcal{Q}_e(s) \le \mathcal{Q}_e(t).$$

The MSE is defined as

$$D = E\left[(X - \mathcal{Q}_d(\pi_n^{-1}(J)))^2\right].$$

The random index J is a function of the source random variable X, the randomness in the channel, and the deterministic functions $\mathcal{Q}_e$ and π_n.

An alternative approach would be to view the quantizer encoder as the composition $\pi_n \cdot \mathcal{Q}_e$ and the quantizer decoder as the composition $\mathcal{Q}_d \cdot \pi_n^{-1}$, by relaxing the previously made monotonicity assumption. This would remove the role of index assignments from the study of quantizers for noisy channels. However, we retain these encoder and decoder decompositions as a convenient way to isolate the effects of index assignments, given known quantizer encoders and decoders.

Assume a binary symmetric channel with bit-error probability ϵ. Denote the probability that index j was received, given that index i was sent by

$$p(j \mid i) = \epsilon^{H_n(i,j)}(1-\epsilon)^{n-H_n(i,j)}$$

for $0 \le \epsilon \le 1/2$, where $H_n(i,j)$ is the Hamming distance between n-bit binary words i and j. Let $q(i|j)$ denote the probability that index i was sent, given that index j was received.

For a given source X, channel $p(\cdot|\cdot)$, index assignment π_n, and quantizer encoder, the quantizer decoder is said to satisfy the *weighted centroid condition* if the codepoints satisfy

$$y_n(j) = \sum_{i=0}^{2^n-1} c_n(i) q\big(\pi_n(i) \mid \pi_n(j)\big).$$

Throughout this paper, we assume a uniform quantizer encoder, so the centroids of the encoder cells are given by

$$c_n(i) = \left(i + \left(\frac{1}{2}\right)\right) 2^{-n}$$

for $0 \le i \le 2^n - 1$. Since the source is uniform and the encoder cells are each of length 2^{-n}, we know that $p(j|i) = q(i|j)$ for all i and j. Hence, the weighted centroid condition implies that

$$\begin{aligned} y_n(j) &= \sum_{i=0}^{2^n-1} c_n(i) p(\pi_n(j) \mid \pi_n(i)) \\ &= \sum_{i=0}^{2^n-1} \frac{i + \left(\frac{1}{2}\right)}{2^n} \epsilon^{H_n(\pi_n(i), \pi_n(j))} \\ &\quad \cdot (1-\epsilon)^{n - H_n(\pi_n(i), \pi_n(j))}. \end{aligned}$$

For a given quantizer encoder and index assignment, we say the quantizer decoder is *channel optimized* if it satisfies the weighted centroid condition.

Notice that if the centroid condition is assumed, then the quantizer decoder $\mathcal{Q}_d$ does not depend on the index assignment, even though the MSE does. In contrast, if the weighted centroid condition is assumed, then the quantizer decoder $\mathcal{Q}_d$ does depend on the index assignment, as does the MSE. Thus, under the centroid condition, minimizing the MSE over all possible index assignments is carried out for a fixed quantizer decoder. However, under the weighted centroid condition, minimizing the MSE over all possible index assignments involves altering the quantizer decoder for each new index assignment.

For any set A, let the indicator function $\mathcal{I}_A(x)$ of A be

$$\mathcal{I}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For each n and each index assignment $\pi_n \in S_{2^n}$, define the function $\lambda_{\pi_n}^{(n)} : [0,1] \to [0,\infty)$ by

$$\lambda_{\pi_n}^{(n)}(x) = \sum_{i=0}^{2^n-1} \frac{1}{2^n \mu_n(i)} \mathcal{I}_{R_n(i)}(x).$$

For a sequence $\pi_n \in S_{2^n}$ (for $n = 1, 2, \dots$) of index assignments, if there exists a function λ such that

$$\lambda(x) = \lim_{n\to\infty} \lambda_{\pi_n}^{(n)}(x)$$

for almost all $x \in [0,1]$ and $\int_0^1 \lambda(x)\,dx = 1$, then λ is said to be a *point density function* with respect to $\{\pi_n\}$.

The following lemma is a result of the fact that index assignments are permutations.

Lemma II.1: For any n and any index assignment $\pi_n \in S_{2^n}$, if $0 \le j \le 2^n - 1$, then

$$\sum_{i=0}^{2^n-1} (1-\epsilon)^{n-H_n(\pi_n(i),\pi_n(j))} \epsilon^{H_n(\pi_n(i),\pi_n(j))} = 1.$$

Let a decoder optimized uniform quantizer denote a rate-n quantizer with a uniform encoder on $[0,1]$ and a channel optimized decoder, along with a uniform source on $[0,1]$, and a binary symmetric channel with bit-error probability ϵ. Let a decoder unoptimized uniform quantizer denote a rate-n uniform quantizer on $[0,1]$, along with a uniform source on $[0,1]$, and a binary symmetric channel with bit-error probability ϵ.

III. Natural Binary Code Index Assignment

For each n, the natural binary code (NBC) is the index assignment defined by

$$\pi_n^{(\mathrm{NBC})}(i) = i \text{ for } 0 \le i \le 2^n - 1.$$

The following lemma is easy to prove and is used in the proof of Proposition III.2.

Lemma III.1:

$$H_{n+1}(i,j) = H_n(i,j), \quad \text{if } 0 \le i,j \le 2^n - 1 \tag{3.1}$$

$$H_{n+1}(i+2^n,j) = H_n(i,j)+1, \quad \text{if } 0 \le i,j \le 2^n - 1 \tag{3.2}$$

$$H_{n+1}(i,j) = H_n(i,j-2^n)+1, \quad \text{if } 0 \le i \le 2^n - 1,\ 2^n \le j \le 2^{n+1}-1 \tag{3.3}$$

$$H_{n+1}(i,j) = H_n(i-2^n,j-2^n), \quad \text{if } 2^n \le i,j \le 2^{n+1}-1. \tag{3.4}$$

Proposition III.2: The codepoints of a decoder optimized uniform quantizer with the NBC index assignment are, for $0 \le j \le 2^n - 1$

$$y_n(j) = \epsilon + (1-2\epsilon)c_n(j). \tag{3.5}$$

Proof: We use induction on n. The weighted centroid condition implies that

$$y_n(j) = 2^{-n-1} \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-H_n(i,j)} \epsilon^{H_n(i,j)} (2i+1). \tag{3.6}$$

In particular, (3.6) gives

$$y_0(0) = \frac{1}{2}$$

which satisfies (3.5). Now assume (3.5) is true for n and consider two cases for $n+1$. If $0 \le j \le 2^n - 1$, then

$$\begin{aligned}
y_{n+1}(j) &= 2^{-n-2} \cdot \sum_{i=0}^{2^{n+1}-1} (1-\epsilon)^{n+1-H_{n+1}(i,j)} \epsilon^{H_{n+1}(i,j)} (2i+1) \\
&= (1-\epsilon)2^{-n-2} \cdot \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-H_n(i,j)} \epsilon^{H_n(i,j)} (2i+1) \\
&\quad + 2^{-n-2} \cdot \sum_{i=2^n}^{2^{n+1}-1} (1-\epsilon)^{n+1-H_{n+1}(i,j)} \epsilon^{H_{n+1}(i,j)} (2i+1) &(3.7)\\
&= \frac{(1-\epsilon)y_n(j)}{2} + 2^{-n-2} \cdot \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-H_n(i,j)} \epsilon^{H_n(i,j)+1} (2i+1+2^{n+1}) &(3.8)\\
&= \frac{(1-\epsilon)y_n(j)}{2} + \frac{\epsilon y_n(j)}{2} + \frac{\epsilon}{2} &(3.9)\\
&= \epsilon + (1-2\epsilon)c_{n+1}(j) &(3.10)
\end{aligned}$$

where the first sum in (3.7) and the second sum in (3.8) follow from (3.1) and (3.2), respectively, (3.9) follows from Lemma II.1, and (3.10) follows from the induction hypothesis.

If $2^n \le j \le 2^{n+1} - 1$, then

$$\begin{aligned}
y_{n+1}(j) &= 2^{-n-2} \cdot \sum_{i=0}^{2^{n+1}-1} (1-\epsilon)^{n+1-H_{n+1}(i,j)} \epsilon^{H_{n+1}(i,j)} (2i+1) \\
&= \epsilon 2^{-n-2} \cdot \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-H_n(i,j-2^n)} \epsilon^{H_n(i,j-2^n)} (2i+1) \\
&\quad + 2^{-n-2} \sum_{i=2^n}^{2^{n+1}-1} (1-\epsilon)^{n+1-H_n(i-2^n,j-2^n)} \epsilon^{H_n(i-2^n,j-2^n)} \cdot (2i+1) &(3.11)\\
&= \frac{y_n(j-2^n)\epsilon}{2} + 2^{-n-2} \cdot \sum_{i=0}^{2^n-1} (1-\epsilon)^{n+1-H_n(i,j-2^n)} \epsilon^{H_n(i,j-2^n)} \cdot (2i+1+2^{n+1})
\end{aligned}$$

$$= \frac{y_n(j-2^n)\epsilon}{2} + \frac{y_n(j-2^n)(1-\epsilon)}{2} + \frac{(1-\epsilon)}{2} \quad (3.12)$$
$$= \epsilon + (1-2\epsilon)c_{n+1}(j) \quad (3.13)$$

where the sums in (3.11) follow from (3.3) and (3.4), respectively, (3.12) follows from Lemma II.1, and (3.13) follows from the induction hypothesis. □

The following theorem shows that with the NBC, the quantizer codepoints are uniformly distributed on a proper subinterval in the source's support region, in the limit of high resolution. As the channel improves (i.e., as $\epsilon \to 0$), the point density function approaches a uniform distribution on $[0,1]$.

Theorem III.3: A sequence of decoder optimized uniform quantizers with the NBC index assignment has a point density function given by

$$\lambda(x) = \begin{cases} \frac{1}{1-2\epsilon} & \text{if } \epsilon < x < 1-\epsilon \\ 0 & \text{otherwise.} \end{cases}$$

Proof: Let

$$\lambda(x) = \begin{cases} \frac{1}{1-2\epsilon} & \text{if } \epsilon < x < 1-\epsilon \\ 0 & \text{if } 0 \le x \le \epsilon \text{ or } (1-\epsilon) \le x \le 1. \end{cases}$$

From (3.5), the codepoints satisfy

$$y_n(j+1) - y_n(j) = (1-2\epsilon)2^{-n}$$

and thus are equally spaced apart. Also,

$$y_n(0) = \epsilon + (1-2\epsilon)\,2^{-n-1}$$
$$y_n(2^n-1) = \epsilon + (1-2\epsilon)\left(1-2^{-n-1}\right).$$

Thus,

$$\mu_n(i) = \begin{cases} (1-2\epsilon)2^{-n} & \text{if } 1 \le i \le 2^n-2 \\ \epsilon + (1-2\epsilon)2^{-n} & \text{if } i \in \{0, 2^n-1\} \end{cases}$$

and, therefore, we get the expression at the bottom of the page. □

IV. Folded Binary Code Index Assignment

For each n, the folded binary code (FBC) is the index assignment defined by

$$\pi_n^{(\text{FBC})}(i) = \begin{cases} 2^{n-1}-1-i & \text{if } 0 \le i \le 2^{n-1}-1 \\ i & \text{if } 2^{n-1} \le i \le 2^n-1. \end{cases}$$

The FBC is closely related to the NBC and has somewhat similar properties for decoder optimized uniform quantizers, as shown by Proposition IV.1 and Theorem IV.2. The proofs of Proposition IV.1 and Theorem IV.2 are similar to those of Proposition III.2 and Theorem III.3, respectively, and are therefore omitted for brevity.

Proposition IV.1: The codepoints of a decoder optimized uniform quantizer with the FBC index assignment are

$$y_n(j) = \begin{cases} \frac{3\epsilon-2\epsilon^2}{2} + (1-2\epsilon)^2 c_n(j) & \text{if } 0 \le j \le 2^{n-1}-1 \\ \frac{5\epsilon-6\epsilon^2}{2} + (1-2\epsilon)^2 c_n(j) & \text{if } 2^{n-1} \le j \le 2^n-1. \end{cases}$$

The following theorem shows that with the FBC, the quantizer codepoints are uniformly distributed on two proper subintervals of the source's support region, in the limit of high resolution. As the channel improves (i.e., as $\epsilon \to 0$), the point density function approaches a uniform distribution on $[0,1]$.

Theorem IV.2: A sequence of decoder optimized uniform quantizers with the FBC index assignment has a point density function given by

$$\lambda(x) = \begin{cases} \frac{1}{(1-2\epsilon)^2} & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

where

$$B = \left(\frac{3\epsilon-2\epsilon^2}{2}, \frac{1-\epsilon+2\epsilon^2}{2}\right) \cup \left(\frac{1+\epsilon-2\epsilon^2}{2}, 1-\left(\frac{3\epsilon-2\epsilon^2}{2}\right)\right).$$

V. Gray Code Index Assignment

For each n, let $\pi_n^{(\text{GC})}$ denote the Gray code (GC) index assignment, recursively defined by

$$\pi_1^{(\text{GC})}(0) = 0$$
$$\pi_1^{(\text{GC})}(1) = 1$$
$$\pi_{n+1}^{(\text{GC})}(i) = \begin{cases} \pi_n^{(\text{GC})}(i) & \text{if } 0 \le i \le 2^n-1 \\ \pi_n^{(\text{GC})}(2^{n+1}-1-i)+2^n & \text{if } 2^n \le i \le 2^{n+1}-1. \end{cases}$$

Define the quantity

$$\hat{H}_n(i,j) = H(\pi_n^{(GC)}(i), \pi_n^{(GC)}(j)).$$

The definition of the GC directly implies the following lemma.

Lemma V.1:

$$\hat{H}_{n+1}(i,j) = \hat{H}_n(i,j) \quad \text{if } 0 \le i,j \le 2^n-1 \quad (5.1)$$
$$\hat{H}_{n+1}(i+2^n,j) = \hat{H}_n(2^n-1-i,j)+1 \quad \text{if } 0 \le i,j \le 2^n-1 \quad (5.2)$$
$$\hat{H}_n(i,j) = \hat{H}_n(2^n-1-i, 2^n-1-j) \quad \text{if } \begin{cases} 2^{n-1} \le j \le 2^n-1 \\ 0 \le i \le 2^n-1. \end{cases} \quad (5.3)$$

$$\lambda_{\pi_n^{(\text{NBC})}}^{(n)}(x) = \begin{cases} \frac{1}{1-2\epsilon} & \text{if } \epsilon+(1-2\epsilon)2^{-n} \le x < (1-\epsilon)-(1-2\epsilon)2^{-n} \\ \frac{1}{2^n\epsilon+(1-2\epsilon)} & \text{if } 0 \le x < \epsilon+(1-2\epsilon)2^{-n} \text{ or } (1-\epsilon)-(1-2\epsilon)2^{-n} \le x \le 1 \end{cases}$$
$$\longrightarrow \lambda(x) \text{ as } n \to \infty.$$

Lemma V.2: The codepoints of a decoder optimized uniform quantizer with the GC index assignment satisfy

$$y_n(j) = 1 - 2^{-n-1} \cdot \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-\hat{H}_n(2^n-1-i,j)} \epsilon^{\hat{H}_n(2^n-1-i,j)} (2i+1)$$

for $0 \le j \le 2^n - 1$.

Proof:

$$\begin{aligned} &2^{-n-1} \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-\hat{H}_n(2^n-1-i,j)} \epsilon^{\hat{H}_n(2^n-1-i,j)} (2i+1) \\ &= \frac{1}{2} + 2^{-n-1} \cdot \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-\hat{H}_n(2^n-1-i,j)} \epsilon^{\hat{H}_n(2^n-1-i,j)} \cdot (2i - (2^n-1)) &\quad (5.4) \\ &= 1 - \left[\frac{1}{2} - 2^{-n-1} \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-\hat{H}_n(i,j)} \epsilon^{\hat{H}_n(i,j)} \cdot (2(2^n-1-i) - (2^n-1)) \right] \\ &= 1 - \left[2^{-n-1} \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-\hat{H}_n(i,j)} \epsilon^{\hat{H}_n(i,j)} (2i+1) \right] \\ &= 1 - y_n(j) &\quad (5.5) \end{aligned}$$

where (5.4) and (5.5) follow from Lemma II.1. □

Corollary V.3: The codepoints of a decoder optimized uniform quantizer with the GC index assignment satisfy

$$y_n(j) = 1 - y_n(2^n - 1 - j)$$

for $2^{n-1} \le j \le 2^n - 1$.

Proof:

$$\begin{aligned} y_n(j) &= 2^{-n-1} \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-\hat{H}_n(i,j)} \epsilon^{\hat{H}_n(i,j)} (2i+1) \\ &= 2^{-n-1} \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-\hat{H}_n(2^n-1-i,2^n-1-j)} \cdot \epsilon^{\hat{H}_n(2^n-1-i,2^n-1-j)} (2i+1) &\quad (5.6) \\ &= 1 - y_n(2^n-1-j) &\quad (5.7) \end{aligned}$$

where (5.6) follows from (5.3), and (5.7) follows from Lemma V.2. □

For $0 \le j \le 2^n - 1$ and $1 \le i \le n$, let $b_n(j,i)$ be the ith most significant bit of the n-bit binary representation of j. Then

$$j = \sum_{i=1}^{n} b_n(j,i) 2^{n-i}$$

and it follows that for $0 \le j \le 2^n - 1$

$$b_n(j,i) = b_{n+1}(j, i+1) = 1 - b_n(2^n - 1 - j, i). \quad (5.8)$$

Proposition V.4: The codepoints of a decoder optimized uniform quantizer with the GC index assignment are

$$y_n(j) = \frac{1}{2} + \frac{1}{2} \sum_{i=1}^{n} (-1)^{b_n(j,i)+1} \left(\frac{1}{2} - \epsilon \right)^i \quad (5.9)$$

for $0 \le j \le 2^n - 1$.

Proof: We use induction on n. The weighted centroid condition implies that for all j

$$y_n(j) = 2^{-n-1} \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-\hat{H}_n(i,j)} \epsilon^{\hat{H}_n(i,j)} (2i+1).$$

For $n = 0$ this reduces to

$$y_0(0) = \frac{1}{2}$$

which satisfies (5.9). Now assume Proposition V.4 is true for n and consider two cases for $n+1$.

If $0 \le j \le 2^n - 1$, then

$$\begin{aligned} &y_{n+1}(j) \\ &= 2^{-n-2} \sum_{i=0}^{2^{n+1}-1} (1-\epsilon)^{n+1-\hat{H}_{n+1}(i,j)} \epsilon^{\hat{H}_{n+1}(i,j)} (2i+1) \\ &= (1-\epsilon) 2^{-n-2} \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-\hat{H}_n(i,j)} \epsilon^{\hat{H}_n(i,j)} (2i+1) \\ &\quad + 2^{-n-2} \sum_{i=0}^{2^n-1} (1-\epsilon)^{n-\hat{H}_n(2^n-1-i,j)} \cdot \epsilon^{\hat{H}_n(2^n-1-i,j)+1} (2i+1+2^{n+1}) &\quad (5.10) \\ &= \frac{(1-\epsilon) y_n(j)}{2} + \frac{\epsilon [2^{n+1} + 2^{n+1}(1 - y_n(j))]}{2^{n+2}} &\quad (5.11) \\ &= \frac{1}{2}\left(\frac{1}{2} + \epsilon\right) + \left(\frac{1}{2} - \epsilon\right) y_n(j) - \frac{1}{2}\left(\frac{1}{2} - \epsilon\right) \\ &= \frac{1}{2}\left(\frac{1}{2} + \epsilon\right) + \left(\frac{1}{2} - \epsilon\right) \left[\frac{1}{2} + \frac{1}{2} \sum_{i=1}^{n} (-1)^{b_n(j,i)+1} \left(\frac{1}{2} - \epsilon\right)^i \right] - \frac{1}{2}\left(\frac{1}{2} - \epsilon\right) &\quad (5.12) \\ &= \frac{1}{2} + \frac{1}{2} \left[-\left(\frac{1}{2} - \epsilon\right) + \sum_{i=2}^{n+1} (-1)^{b_{n+1}(j,i)+1} \left(\frac{1}{2} - \epsilon\right)^i \right] &\quad (5.13) \\ &= \frac{1}{2} + \frac{1}{2} \sum_{i=1}^{n+1} (-1)^{b_{n+1}(j,i)+1} \left(\frac{1}{2} - \epsilon\right)^i &\quad (5.14) \end{aligned}$$

where the sums in (5.10) follow from (5.1) and (5.2), respectively, (5.11) follows from Lemmas II.1 and V.2, (5.12) follows from the induction hypothesis, (5.13) follows from (5.8), and (5.14) follows from the fact that $b_{n+1}(j,1)=0$ whenever $0 \le j \le 2^n - 1$.

If $2^n \le j \le 2^{n+1}-1$, then

$$\begin{aligned} & y_{n+1}(j) \\ &= 1 - \left(\frac{1}{2} + \frac{1}{2}\sum_{i=1}^{n+1}(-1)^{b_{n+1}(2^{n+1}-1-j,i)+1} \cdot \left(\frac{1}{2}-\epsilon\right)^i\right) \quad (5.15) \\ &= \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{n+1}(-1)^{b_{n+1}(j,i)+1}\left(\frac{1}{2}-\epsilon\right)^i \quad (5.16) \end{aligned}$$

where (5.15) follows from Corollary V.3 and (5.14), and (5.16) follows from (5.8). □

To show that no point density function arises from the GC index assignment, we will show that

$$\lambda(x) = \lim_{n\to\infty} \lambda^{(n)}_{\pi_n^{(GC)}}(x) = 0$$

almost everywhere on $[0,1]$, and hence

$$\int_0^1 \lambda(x)dx \neq 1.$$

To simplify notation, let $\lambda^{(n)}_{\pi_n^{(\mathrm{GC})}}$ be denoted by λ_n.

First, several preliminary results are necessary. In order to determine the asymptotic behavior of λ_n we examine the values of $\mu_n(i)$ and the relationship of $R_n(i)$ to $R_{n-1}(\lfloor i/2 \rfloor)$. For any fixed value of n there are groups of nearest neighbor cells with the same length. These groups and the properties of the cells in them are key to the subsequent results.

Lemma V.6 describes each of these groups by the number of cells in the group and their common length. This is done by identifying a cell in each group whose index is of the form $i = 2^{n-k}-1$ and considering its length. Lemma V.5 shows that the codepoints are indexed in increasing order, and is used in the proof of Lemma V.6.

Lemma V.5: The codepoints of a decoder optimized uniform quantizer with the GC index assignment satisfy $y_n(j+1) > y_n(j)$ whenever $0 \le j \le 2^n - 2$.

Proof: Let

$$k = \min\{k' : b_n(j,i) = 1,\ \forall\, i \ge k'\}.$$

Then the binary representation of j ends in exactly $n-k+1$ ones, and therefore,

$$\begin{aligned} b_n(j,k-1) &= 0 & \\ b_n(j,i) &= 1 & \text{for } i \ge k \\ b_n(j+1,i) &= b_n(j,i) & \text{for } 1 \le i \le k-2 \\ b_n(j+1,k-1) &= 1 & \\ b_n(j+1,i) &= 0 & \text{for } i \ge k. \end{aligned}$$

Thus, from (5.9), we have

$$\begin{aligned} & y_n(j+1) - y_n(j) \\ &= \frac{1}{2}\sum_{i=1}^{n}\left[(-1)^{b_n(j,i)} - (-1)^{b_n(j+1,i)}\right]\left(\frac{1}{2}-\epsilon\right)^i \\ &= \frac{1}{2}\sum_{i=k}^{n}\left[(-1)^1 - (-1)^0\right]\left(\frac{1}{2}-\epsilon\right)^i \\ &\quad + \frac{1}{2}\left[(-1)^0 - (-1)^1\right]\left(\frac{1}{2}-\epsilon\right)^{k-1} \\ &= \left(\frac{1}{2}-\epsilon\right)^{k-1} - \sum_{i=k}^{n}\left(\frac{1}{2}-\epsilon\right)^i \\ &> 0. \end{aligned}$$

□

Lemma V.6: For $1 \le k \le n-1$, a decoder optimized uniform quantizer with the GC index assignment has 2^k nearest neighbor cells whose lengths equal $\mu_n(2^{n-k}-1)$.

Proof: By Lemma V.5, the codepoints $y_n(j)$ are increasing in j. Thus, for $1 \le i \le 2^n - 2$

$$\mu_n(i) = \frac{1}{2}\big(y_n(i+1) - y_n(i-1)\big).$$

Note that for $1 \le k \le n-1$, the binary representation of 2^{n-k} is

$$\underbrace{00\ldots01}_{k}\underbrace{00\ldots00}_{n-k}$$

and the binary representation of $2^{n-k}-2$ is

$$\underbrace{00\ldots00}_{k}\underbrace{11\ldots10}_{n-k},$$

which agree on the first $k-1$ digits and on the last digit. By (5.9), the difference between the ith and jth codepoints depends only on the locations in the binary representations of i and j where they differ. For all $w \in \{0,\ldots,2^{k-1}-1\}$ and $b \in \{0,1\}$, the binary representations of

$$2^{n-k} + w2^{n-k+1} + b$$

and

$$2^{n-k} - 2 + w2^{n-k+1} + b$$

agree in exactly the same locations that 2^{n-k} and $2^{n-k}-2$ agree in, and hence,

$$\begin{aligned} \mu_n(2^{n-k}-1) &= \frac{1}{2}\left(y_n(2^{n-k}) - y_n(2^{n-k}-2)\right) \\ &= \frac{1}{2}\big(y_n(2^{n-k} + w2^{n-k+1} + b) \\ &\qquad - y_n(2^{n-k} - 2 + w2^{n-k+1} + b)\big). \end{aligned}$$

The claimed 2^k nearest neighbor cells are thus,

$$R(2^{n-k} - 1 + w2^{n-k+1} + b).$$

□

The next lemma computes $\mu_n(i)$ for $0 \le i \le 2^n - 1$. By Lemma V.6, it suffices to consider the lengths of $R_n(0)$, $R_n(2^n-1)$, and $R_n(2^{n-k}-1)$ for $1 \le k \le n-1$.

Lemma V.7: For a decoder optimized uniform quantizer with the GC index assignment

$$\mu_n(0) = \mu_n(2^n-1) = \frac{\epsilon + \frac{1}{2}(\frac{1}{2}-\epsilon)^n}{\frac{1}{2}+\epsilon} \quad (5.17)$$

and for $1 \leq k \leq n-1$

$$\mu_n(2^{n-k}-1) = \frac{\epsilon(\frac{1}{2}-\epsilon)^k + \frac{1}{2}(\frac{1}{2}-\epsilon)^n}{\frac{1}{2}+\epsilon}.$$

Proof: By Corollary V.3, Lemma V.5, and the definitions of $R_n(0)$ and $R_n(2^n-1)$

$$\begin{aligned}\mu_n(2^n-1) &= 1 - \frac{1}{2}(y_n(2^n-2) + y_n(2^n-1)) \\ &= 1 - \frac{1}{2}(1 - y_n(1) + 1 - y_n(0)) \\ &= \mu_n(0).\end{aligned}$$

Since the n-bit binary representations of 0 and 1 differ only in the least significant bit

$$\begin{aligned}\mu_n(0) &= \frac{1}{2}(y_n(0) + y_n(1)) \\ &= \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{n-1}(-1)^{b_n(0,i)+1}\left(\frac{1}{2}-\epsilon\right)^i \\ &= \frac{1}{2} - \frac{1}{2}\left(\frac{1-(\frac{1}{2}-\epsilon)^n}{\frac{1}{2}+\epsilon} - 1\right) \\ &= \frac{\epsilon + \frac{1}{2}(\frac{1}{2}-\epsilon)^n}{\frac{1}{2}+\epsilon} \qquad (5.18)\end{aligned}$$

where (5.18) follows from (5.9). Recall from the proof of Lemma V.6 that

$$\mu_n(2^{n-k}-1) = \frac{1}{2}(y_n(2^{n-k}) - y_n(2^{n-k}-2))$$

and that the binary representations of 2^{n-k} and $2^{n-k}-2$ are

$$\underbrace{00\ldots01}_{k}\underbrace{00\ldots00}_{n-k}$$

and

$$\underbrace{00\ldots00}_{k}\underbrace{11\ldots10}_{n-k}$$

respectively. Combining this information with (5.9) gives

$$\begin{aligned}&\mu_n(2^{n-k}-1) \\ &= \frac{1}{2}(y_n(2^{n-k}) - y_n(2^{n-k}-2)) \\ &= \frac{1}{4}\sum_{i=1}^{n}(-1)^{b_n(2^{n-k},i)+1}\left(\frac{1}{2}-\epsilon\right)^i \\ &\quad -\frac{1}{4}\sum_{i=1}^{n}(-1)^{b_n(2^{n-k}-2,i)+1}\left(\frac{1}{2}-\epsilon\right)^i \\ &= \frac{1}{2}\left(\frac{1}{2}-\epsilon\right)^k - \frac{1}{2}\sum_{i=k+1}^{n-1}\left(\frac{1}{2}-\epsilon\right)^i \\ &= \frac{\epsilon(\frac{1}{2}-\epsilon)^k + \frac{1}{2}(\frac{1}{2}-\epsilon)^n}{\frac{1}{2}+\epsilon}. \qquad \square\end{aligned}$$

The next result follows directly from Lemma V.7 and will be important in determining the behavior of λ_n as $n \to \infty$.

Corollary V.8: For a decoder optimized uniform quantizer with the GC index assignment

$$\lim_{n\to\infty}\frac{1}{2^n\mu_n(0)} = \lim_{n\to\infty}\frac{1}{2^n\mu_n(2^n-1)} = 0$$

and for each fixed $k \geq 1$

$$\lim_{n\to\infty}\frac{1}{2^n\mu_n(2^{n-k}-1)} = 0. \qquad (5.19)$$

Define the sets

$$\begin{aligned}E_{n,k} &= \bigcup_{i:\mu_n(i)=\mu_n(2^{n-k}-1)} R_n(i) \quad \text{for } 1 \leq k \leq n-1 \\ F_n &= R_n(0) \cup R_n(2^n-1) \qquad \text{for } n \geq 1\end{aligned}$$

and note that $E_{n,k}$ and F_n are disjoint for all k and n.

Lemma V.9: For a decoder optimized uniform quantizer with the GC index assignment

i) $R_1(0) \supset R_2(0) \supset R_3(0) \supset \cdots$

ii) $R_1(2^1-1) \supset R_2(2^2-1) \supset R_3(2^3-1) \supset \cdots$

iii) $F_1 \supset F_2 \supset F_3 \supset \cdots$

iv) $\forall\, k \geq 1, E_{k+1,k} \supset E_{k+2,k} \supset E_{k+3,k} \supset \cdots$

v) $m\left(\bigcap_{n=k+1}^{\infty} E_{n,k}\right) = \frac{\epsilon(1-2\epsilon)^k}{\frac{1}{2}+\epsilon}$.

Proof: By Lemma V.7

$$\frac{y_n(0)+y_n(1)}{2} = \frac{\epsilon + \frac{1}{2}(\frac{1}{2}-\epsilon)^n}{\frac{1}{2}+\epsilon}$$

which is decreasing in n. This proves part i) and also shows that (using Corollary V.3)

$$\frac{y_n(2^n-2)+y_n(2^n-1)}{2} = \frac{1-y_n(1)+1-y_n(0)}{2}$$

is increasing in n, thus proving part ii). Part iii) follows directly from parts i) and ii).

To prove part iv), first note that (5.9) implies that for $0 \leq i \leq 2^{n-1}-1$

$$\begin{aligned}y_n(2i) &= y_{n-1}(i) - \frac{1}{2}\left(\frac{1}{2}-\epsilon\right)^n \\ y_n(2i+1) &= y_{n-1}(i) + \frac{1}{2}\left(\frac{1}{2}-\epsilon\right)^n.\end{aligned}$$

Also assume without loss of generality that

$$\{x : |y_n(i) - x| = |y_n(i+1) - x|\} \subseteq R_n(i+1).$$

Suppose $1 \leq i \leq 2^n-2$ and $n \geq 2$.

If i is even (say $i = 2j$), then

$$\begin{aligned}R_n(i) &= R_n(2j) \\ &= \left[\frac{y_n(2(j-1)+1) + y_n(2j)}{2}, \frac{y_n(2j)+y_n(2j+1)}{2}\right) \\ &= \left[\frac{y_{n-1}(j-1)+y_{n-1}(j)}{2}, y_{n-1}(j)\right) \\ &\subset R_{n-1}(j) = R_{n-1}\left(\frac{i}{2}\right) \qquad (5.20)\end{aligned}$$

where (5.20) follows from the definition of $R_{n-1}(i)$.

If i is odd (say $i = 2j+1$), then

$$\begin{aligned} R_n(i) &= R_n(2j+1) \\ &= \left[\frac{y_n(2j)+y_n(2j+1)}{2}, \right. \\ &\qquad \left. \frac{y_n(2j+1)+y_n(2j+2)}{2}\right) \\ &= \left[y_{n-1}(j), \frac{y_{n-1}(j)+y_{n-1}(j+1)}{2}\right) \\ &\subset R_{n-1}(j) = R_{n-1}\left(\frac{(i-1)}{2}\right) \qquad (5.21) \end{aligned}$$

where (5.21) follows from the definition of $R_{n-1}(i)$.

For each cell $R_n(i)$ in $E_{n,k}$ with $1 \le k \le n-1$, the proof of Lemma V.6 shows that i is of the form

$$i = 2^{n-k} - 1 + w2^{n-k+1} + b$$

where $w \in \{0, \dots, 2^{k-1}-1\}$ and $b \in \{0,1\}$. If $R_n(i) \subset E_{n,k}$ and i is even, then $b = 1$ and

$$i = 2^{n-k} + w2^{n-k+1}$$

or equivalently

$$\frac{i}{2} = 2^{(n-1)-k} + w2^{(n-1)-k+1}$$

which implies $R_{n-1}(i/2) \subset E_{n-1,k}$. Equation (5.20) shows that $R_n(i) \subset R_{n-1}(i/2)$, and hence, $R_n(i) \subset E_{n-1,k}$.

Likewise, if $R_n(i) \subset E_{n,k}$ and i is odd, then $b = 0$ and

$$i = 2^{n-k} - 1 + w2^{n-k+1}$$

or equivalently

$$\frac{(i-1)}{2} = 2^{(n-1)-k} - 1 + w2^{(n-1)-k+1}$$

which implies $R_{n-1}((i-1)/2) \subset E_{n-1,k}$. Equation (5.21) shows that $R_n(i) \subset R_{n-1}((i-1)/2)$, and hence, $R_n(i) \subset E_{n-1,k}$. Therefore,

$$E_{n,k} = \bigcup_{i:\mu_n(i)=\mu_n(2^{n-k}-1)} R_n(i) \subset E_{n-1,k}$$

proving part iv).

Since $\{E_{n,k}\}_{n=k+1}^{\infty}$ is a decreasing sequence of bounded sets (for each fixed k) by part iv)

$$\begin{aligned} m\left(\bigcap_{n=k+1}^{\infty} E_{n,k}\right) &= \lim_{n\to\infty} m(E_{n,k}) \\ &= \lim_{n\to\infty} m\left(\bigcup_{i:\mu_n(i)=\mu_n(2^{n-k}-1)} R_n(i)\right) \\ &= \lim_{n\to\infty} \sum_{i:\mu_n(i)=\mu_n(2^{n-k}-1)} m(R_n(i)) \\ &= \lim_{n\to\infty} 2^k \left(\frac{\epsilon(\frac{1}{2}-\epsilon)^k + \frac{1}{2}(\frac{1}{2}-\epsilon)^n}{\frac{1}{2}+\epsilon}\right) \qquad (5.22) \\ &= \frac{\epsilon(1-2\epsilon)^k}{\frac{1}{2}+\epsilon} \end{aligned}$$

where (5.22) follows from Lemmas V.6 and V.7. This proves part v). □

The following theorem shows that the sequence of functions $\lambda^{(n)}_{\pi_n^{(\mathrm{GC})}}$ does not converge to a point density function as $n \to \infty$.

Theorem V.10: A sequence of decoder optimized uniform quantizers with the GC index assignment does not have a point density function.

Proof: We construct disjoint sets $E_k \subset [0,1]$ whose union has measure 1 and for which $\lim_{n\to\infty} \lambda_n(x) = 0$ for all $x \in E_k$ and for all k.

Let $E_0 = \bigcap_{n=1}^{\infty} F_n$. Then for any n and any $x \in E_0$, either $x \in R_n(0)$ or $x \in R_n(2^n - 1)$, and therefore,

$$\lambda_n(x) = \frac{1}{2^n \mu_n(0)} = \frac{1}{2^n \mu_n(2^n-1)}$$

by Lemma V.7. Hence, for any $x \in E_0$

$$\lim_{n\to\infty} \lambda_n(x) = \lim_{n\to\infty} \frac{1}{2^n \mu_n(0)} = 0$$

by Corollary V.8.

For $k \ge 1$, let

$$E_k = \bigcap_{n=k+1}^{\infty} E_{n,k}.$$

Then for any n and k such that $n \ge k+1$ and for any $x \in E_k$, there exists an i such that $x \in R_n(i)$ and $\mu_n(i) = \mu_n(2^{n-k}-1)$, which implies

$$\lambda_n(x) = \frac{1}{2^n \mu_n(2^{n-k}-1)}.$$

Hence, for any $x \in E_k$

$$\lim_{n\to\infty} \lambda_n(x) = \lim_{n\to\infty} \frac{1}{2^n \mu_n(2^{n-k}-1)} = 0$$

by Corollary V.8.

Part v) of Lemma V.9 shows that E_k is nonempty for all $k \ge 1$. Subsequently, it will be shown that E_0 is nonempty.

E_0 and E_k are disjoint for all $k \ge 1$, since $E_{n,k}$ and F_n are disjoint for all k and n. The sets E_k are disjoint for $k \ge 1$, for otherwise $E_{n,i}$ and $E_{n,j}$ would intersect for some n and some $i \ne j$. Therefore,

$$\begin{aligned} m\left(\bigcup_{k=0}^{\infty} E_k\right) &= \sum_{k=0}^{\infty} m(E_k) \\ &= m\left(\bigcap_{n=1}^{\infty} F_n\right) + \sum_{k=1}^{\infty} m\left(\bigcap_{n=k+1}^{\infty} E_{n,k}\right) \\ &= \lim_{n\to\infty} m(F_n) + \sum_{k=1}^{\infty} \frac{\epsilon(1-2\epsilon)^k}{\frac{1}{2}+\epsilon} \qquad (5.23) \\ &= \frac{2\epsilon}{\frac{1}{2}+\epsilon} + \frac{\epsilon}{\frac{1}{2}+\epsilon}\left(\frac{1}{2\epsilon} - 1\right) \qquad (5.24) \\ &= 1 \end{aligned}$$

where the first term in (5.23) follows from Lemma V.9, part iii) and the boundedness of $R_n(0)$ and $R_n(2^n-1)$, the second term in (5.23) follows from Lemma V.9, part v), and the first term in (5.24) follows from (5.17). Thus, the set

$$\left\{x \in [0,1] : \lim_{n\to\infty} \lambda_n(x) \ne 0\right\}$$

has measure 0 since it is a subset of

$$\left(\bigcup_{k=0}^{\infty} E_k\right)^c \cap [0,1].$$

□

VI. Randomly Chosen Index Assignments

Suppose for each $n \geq 1$ an index assignment Π_n is chosen uniformly at random from the set of all $2^n!$ index assignments. Then λ does not exist in a deterministic sense as the limit of $\lambda_{\Pi_n}^{(n)}$. However, the distribution of codepoints can still be characterized probabilistically.

Proposition VI.1: Suppose an index assignment is chosen uniformly at random for a decoder optimized uniform quantizer. Then for all j, the expected value of the jth codepoint is

$$E[y_n(j)] = \frac{1}{2} + \left(c_n(j) - \frac{1}{2}\right)\left(\frac{1}{1-2^{-n}}\right)\left((1-\epsilon)^n - 2^{-n}\right).$$

Proof: Let $\delta = \epsilon/(1-\epsilon)$ and note that $(1-\epsilon)(1+\delta) = 1$. Then

$$\begin{aligned} E[y_n(j)] &= \sum_{i=0}^{2^n-1} c_n(i) E[p(\Pi_n(j)|\Pi_n(i))] \\ &= \sum_{i=0}^{2^n-1} c_n(i) \cdot \frac{1}{2^n!} \cdot \sum_{\pi_n \in S_{2^n}} \epsilon^{H_n(\pi_n(i),\pi_n(j))}(1-\epsilon)^{n-H_n(\pi_n(i),\pi_n(j))} \\ &= \frac{(1-\epsilon)^n}{2^n!} \sum_{i=0}^{2^n-1} c_n(i) \sum_{\pi_n \in S_{2^n}} \delta^{H_n(\pi_n(i),\pi_n(j))} \\ &= \frac{(1-\epsilon)^n}{2^n!}\Bigg(2^n! c_n(j) + \sum_{i\neq j} c_n(i) \sum_{\pi_n \in S_{2^n}} \delta^{H_n(\pi_n(i),\pi_n(j))}\Bigg) \\ &= (1-\epsilon)^n \left(c_n(j) + \left(\sum_{i=0}^{2^n-1} c_n(i) - c_n(j)\right) \cdot \left(\frac{2^n(2^n-2)!}{2^n!}\sum_{k=1}^{n}\binom{n}{k}\delta^k\right)\right) \quad (6.1) \\ &= (1-\epsilon)^n \left(c_n(j) + \left(2^{n-1} - c_n(j)\right)\frac{(1+\delta)^n - 1}{(2^n-1)}\right) \quad (6.2) \\ &= \frac{1}{2} + \left(c_n(j) - \frac{1}{2}\right)\left(\frac{1}{1-2^{-n}}\right)\left((1-\epsilon)^n - 2^{-n}\right). \end{aligned}$$

To justify (6.1), consider the following observations. Suppose $i \neq j$. There are 2^n possible values $\pi_n(j)$ can have, and for each one there are $2^n - 1$ values $\pi_n(i)$ can take, $\binom{n}{k}$ of which must have Hamming distance k from $\pi_n(j)$. Given any of the $2^n(2^n - 1)$ possible choices of $\pi_n(j)$ and $\pi_n(i)$, there are $(2^n-2)!$ ways to assign the remaining index assignment words. Equation (6.2) follows from the fact that

$$\sum_{i=0}^{2^n-1} c_n(i) = 2^{-n}\sum_{i=0}^{2^n-1}\left(i + \left(\frac{1}{2}\right)\right) = 2^{n-1}. \qquad \square$$

With Proposition VI.1, the variance of the jth codepoint is

$$\begin{aligned} \mathrm{Var}(y_n(j)) &= \mathrm{Var}\left(y_n(j) - \frac{1}{2}\right) \\ &= E\left[\left(y_n(j) - \frac{1}{2}\right)^2\right] - \left[\left(c_n(j) - \frac{1}{2}\right)\left(\frac{1}{1-2^{-n}}\right)\left((1-\epsilon)^n - 2^{-n}\right)\right]^2. \end{aligned} \quad (6.3)$$

The motivation for the form of (6.3) will become clear in the proof of Theorem VI.4. Evaluation of the expectation in (6.3) yields Proposition VI.3.

Lemma VI.2:

$$\sum_{i=0}^{2^n-1}\left(c_n(i) - \frac{1}{2}\right)^2 = \frac{1}{12}\left(2^n - 2^{-n}\right).$$

Proof:

$$\begin{aligned} &\sum_{i=0}^{2^n-1}\left(c_n(i) - \frac{1}{2}\right)^2 \\ &= \sum_{i=0}^{2^n-1}\left(2^{-n}\left(i + \frac{1}{2}\right) - \frac{1}{2}\right)^2 \\ &= 2^{-2n}\left[\sum_{i=0}^{2^n-1} i^2 - (2^n-1)\sum_{i=0}^{2^n-1} i + 2^n\left(\frac{2^n-1}{2}\right)^2\right] \\ &= 2^{-2n}\left[\frac{2^n(2^n-1)(2^{n+1}-1)}{6} - (2^n-1)\frac{2^n(2^n-1)}{2} + 2^n\left(\frac{2^n-1}{2}\right)^2\right] \\ &= \frac{1}{12}\left(2^n - 2^{-n}\right). \end{aligned} \qquad \square$$

Proposition VI.3: Suppose for each n, an index assignment is chosen uniformly at random for the nth quantizer (of rate n) in a sequence of decoder optimized uniform quantizers. Then for all j, the variance of the jth codepoint decays to zero at the rate $\mathrm{Var}(y_n(j)) = O(2^{-\beta n})$ as $n \to \infty$, where $\beta = -\log_2(1 - 2\epsilon + 2\epsilon^2)$.

Proof: Recall from (6.3) that the variance of $y_n(j)$ is

$$\begin{aligned} &\mathrm{Var}(y_n(j)) \\ &= \mathrm{Var}\left(y_n(j) - \frac{1}{2}\right) \\ &= E\left[\left(y_n(j) - \frac{1}{2}\right)^2\right] - \left[\left(c_n(j) - \frac{1}{2}\right)\left(\frac{1}{1-2^{-n}}\right)\left((1-\epsilon)^n - 2^{-n}\right)\right]^2 \end{aligned} \quad (6.4)$$

whose second term goes to zero as $O((1-\epsilon)^{2n})$ when $n \to \infty$. Expanding the first term of (6.4) yields

$$\begin{aligned}
&E\left[\left(y_n(j)-\frac{1}{2}\right)^2\right]\\
&= E\left[\left(\sum_{i=0}^{2^n-1}\left(c_n(i)-\frac{1}{2}\right)p(\Pi_n(j)|\Pi_n(i))\right)^2\right]\\
&= \sum_{i=0}^{2^n-1}\sum_{l=0}^{2^n-1}\left(c_n(i)-\frac{1}{2}\right)\left(c_n(l)-\frac{1}{2}\right)\\
&\quad\cdot E[p(\Pi_n(j)|\Pi_n(i))p(\Pi_n(j)|\Pi_n(l))]\\
&= (1-\epsilon)^{2n}\sum_{i=0}^{2^n-1}\sum_{l=0}^{2^n-1}\left(c_n(i)-\frac{1}{2}\right)\left(c_n(l)-\frac{1}{2}\right)\\
&\quad\cdot E\left[\delta^{H_n(\Pi_n(i),\Pi_n(j))+H_n(\Pi_n(l),\Pi_n(j))}\right].
\end{aligned} \tag{6.5}$$

We consider four cases. The computation in the last three cases is justified by an argument similar to the one used to justify (6.1).

1) If $i = l = j$, then

$$E\left[\delta^{H_n(\Pi_n(i),\Pi_n(j))+H_n(\Pi_n(l),\Pi_n(j))}\right] = 1.$$

2) If $i = l \neq j$, then

$$\begin{aligned}
&E\left[\delta^{H_n(\Pi_n(i),\Pi_n(j))+H_n(\Pi_n(l),\Pi_n(j))}\right]\\
&= \frac{1}{2^n!}\sum_{\pi_n\in S_{2^n}}\delta^{2H_n(\pi_n(i),\pi_n(j))}\\
&= \frac{2^n(2^n-2)!}{2^n!}\sum_{r=1}^{n}\delta^{2r}\binom{n}{r}\\
&= \frac{(1+\delta^2)^n-1}{2^n-1}.
\end{aligned}$$

3) If $i \neq l$, $i \neq j$, and $l \neq j$, then

$$\begin{aligned}
&E\left[\delta^{H_n(\Pi_n(i),\Pi_n(j))+H_n(\Pi_n(l),\Pi_n(j))}\right]\\
&= \frac{1}{2^n!}\sum_{\pi_n\in S_{2^n}}\delta^{H_n(\pi_n(i),\pi_n(j))+H_n(\pi_n(l),\pi_n(j))}\\
&= \frac{2^n(2^n-3)!}{2^n!}\sum_{k=1}^{n}\sum_{m=1}^{n}\delta^{k+m}\binom{n}{k}\\
&\quad\cdot\begin{cases}\binom{n}{m} & \text{if } m\neq k\\ \binom{n}{m}-1 & \text{if } m=k\end{cases}\\
&= \frac{1}{(2^n-1)(2^n-2)}\\
&\quad\cdot\left\{\left[\sum_{k=1}^{n}\binom{n}{k}\delta^k\right]^2-\sum_{k=1}^{n}\binom{n}{k}\delta^{2k}\right\}\\
&= \frac{((1+\delta)^n-1)^2-((1+\delta^2)^n-1)}{(2^n-1)(2^n-2)}\\
&= \frac{2+(1+\delta)^{2n}-2(1+\delta)^n-(1+\delta^2)^n}{(2^n-1)(2^n-2)}.
\end{aligned}$$

4) If $j = i \neq l$ (or $j = l \neq i$), then

$$\begin{aligned}
&E\left[\delta^{H_n(\Pi_n(i),\Pi_n(j))+H_n(\Pi_n(l),\Pi_n(j))}\right]\\
&= \frac{1}{2^n!}\sum_{\pi_n\in S_{2^n}}\delta^{H_n(\pi_n(l),\pi_n(j))}\\
&= \frac{2^n(2^n-2)}{2^n!}\sum_{r=1}^{n}\delta^r\binom{n}{r}\\
&= \frac{(1+\delta)^n-1}{2^n-1}.
\end{aligned}$$

Thus, (6.5) can be written in terms of the four cases as

$$\begin{aligned}
&E\left[\left(y_n(j)-\frac{1}{2}\right)^2\right]\\
&= (1-\epsilon)^{2n}\left(c_n(j)-\frac{1}{2}\right)^2\\
&\quad+(1-\epsilon)^{2n}\left(\frac{(1+\delta^2)^n-1}{2^n-1}\right)\\
&\quad\cdot\left[\left(\sum_{i=0}^{2^n-1}\left(c_n(i)-\frac{1}{2}\right)^2\right)-\left(c_n(j)-\frac{1}{2}\right)^2\right]+(1-\epsilon)^{2n}\\
&\quad\cdot\left(\frac{2+(1+\delta)^{2n}-2(1+\delta)^n-(1+\delta^2)^n}{(2^n-1)(2^n-2)}\right)\\
&\quad\cdot\sum_{i\neq j}\sum_{l\neq i,l\neq j}\left(c_n(i)-\frac{1}{2}\right)\left(c_n(l)-\frac{1}{2}\right)\\
&\quad+(1-\epsilon)^{2n}\cdot 2\left(\frac{(1+\delta)^n-1}{2^n-1}\right)\\
&\quad\cdot\sum_{l\neq j}\left(c_n(j)-\frac{1}{2}\right)\left(c_n(l)-\frac{1}{2}\right).
\end{aligned} \tag{6.6}$$

The first term in (6.6) decays to 0 as $O((1-\epsilon)^{2n})$ as $n \to \infty$. The second term in (6.6) is

$$\begin{aligned}
&(1-\epsilon)^{2n}\left(\frac{(1+\delta^2)^n-1}{2^n-1}\right)^n\\
&\quad\cdot\left[\left(\sum_{i=0}^{2^n-1}\left(c_n(i)-\frac{1}{2}\right)^2\right)-\left(c_n(j)-\frac{1}{2}\right)^2\right]\\
&= (1-\epsilon)^{2n}\left(\frac{(1+\delta^2)^n-1}{2^n-1}\right)\\
&\quad\cdot\left[\frac{2^{2n}-1}{12\cdot 2^n}-\left(c_n(j)-\frac{1}{2}\right)^2\right]\\
&= \left[(1-2\epsilon+2\epsilon^2)^n-(1-\epsilon)^{2n}\right]\\
&\quad\cdot\left[\frac{2^n+1}{12\cdot 2^n}+\frac{j^2}{2^{2n}(2^n-1)}+\frac{2^n-1-4j}{2^{2n+2}}\right]\\
&= O\left((1-2\epsilon+2\epsilon^2)^n\right)
\end{aligned}$$

as $n \to \infty$, since $0 < \epsilon < 1$. To evaluate the third term in (6.6) note that

$$\begin{aligned}
&\sum_{i\neq j}\sum_{l\neq i,l\neq j}\left(c_n(i)-\frac{1}{2}\right)\left(c_n(l)-\frac{1}{2}\right)\\
&= \left[\sum_{i=0}^{2^n-1}\left(c_n(i)-\frac{1}{2}\right)\right]^2-\left[\sum_{i=0}^{2^n-1}\left(c_n(i)-\frac{1}{2}\right)^2\right]
\end{aligned}$$

$$-2\left(c_n(j)-\frac{1}{2}\right)\cdot\left[-\left(c_n(j)-\frac{1}{2}\right)+\sum_{i=0}^{2^n-1}\left(c_n(i)-\frac{1}{2}\right)\right]$$
$$= 0-\frac{1}{12}\left(2^n-2^{-n}\right)+2\left(c_n(j)-\frac{1}{2}\right)^2.$$

Thus, since $(1-\epsilon)(1+\delta)=1$, the third term in (6.6) is

$$\left(\frac{2(1-\epsilon)^{2n}+1-2(1-\epsilon)^n-(1-2\epsilon+2\epsilon^2)^n}{(2^n-1)(2^n-2)}\right)\cdot\left(2\left(c_n(j)-\frac{1}{2}\right)^2-\frac{2^n}{12}+\frac{1}{12\cdot 2^n}\right)$$

which tends to 0 as $O(2^{-n})$ as $n\to\infty$. To evaluate the forth term in (6.6) note that

$$\sum_{l\neq j}\left(c_n(j)-\frac{1}{2}\right)\left(c_n(l)-\frac{1}{2}\right) = \left(c_n(j)-\frac{1}{2}\right)\sum_{l\neq j}\left(c_n(l)-\frac{1}{2}\right) = -\left(c_n(j)-\frac{1}{2}\right)^2.$$

Thus, the forth term in (6.6) is

$$(1-\epsilon)^{2n}\cdot 2\left(\frac{(1+\delta)^n-1}{2^n-1}\right)\cdot\sum_{l\neq j}\left(c_n(j)-\frac{1}{2}\right)\left(c_n(l)-\frac{1}{2}\right)$$
$$= -2\left(\frac{(1-\epsilon)^n-(1-\epsilon)^{2n}}{2^n-1}\right)\left(c_n(j)-\frac{1}{2}\right)^2 = O\left(2^{-n}\right)$$

as $n\to\infty$. Thus,

$$\mathrm{Var}(y_n(j)) = O\left(\left(1-2\epsilon+2\epsilon^2\right)^n\right)=O(2^{-\beta n})$$

as $n\to\infty$, where $\beta=-\log_2(1-2\epsilon+2\epsilon^2)>0$. □

Proposition VI.3 is key to the proof of the next result. The following theorem shows that asymptotically, an arbitrarily large fraction of index assignments induce an arbitrarily large fraction of codepoints to be arbitrarily close to $1/2$. This result is in contrast to the fact that the NBC index assignment has an arbitrarily small fraction of codepoints arbitrarily close to $1/2$.

Theorem VI.4: For a decoder optimized uniform quantizer, arbitrarily small $r, s, t>0$, and n sufficiently large, at least $(1-r)2^n!$ index assignments each have at least $(1-s)2^n$ codepoints within a distance of t from $1/2$.

Proof: Assume Π_n is chosen uniformly at random from the set S_{2^n} of all $2^n!$ index assignments. Let

$$\delta=\frac{\epsilon}{1-\epsilon}$$

and note that $(1-\epsilon)(1+\delta)=1$. Also, let

$$a_n=\left(c_n(j)-\frac{1}{2}\right)\left(\frac{1}{1-2^{-n}}\right)\left((1-\epsilon)^n-2^{-n}\right).$$

By the Chebychev inequality, for any $t>0$

$$\begin{aligned}P\left[\left|y_n(j)-\frac{1}{2}\right|>t\right] &= P\left[\left|y_n(j)-\frac{1}{2}-a_n+a_n\right|>t\right]\\ &\le P\left[\left|y_n(j)-\frac{1}{2}-a_n\right|>t-|a_n|\right]\\ &= P\left[|y_n(j)-E[y_n(j)]|>t-|a_n|\right]\\ &< \frac{\mathrm{Var}(y_n(j))}{(t-|a_n|)^2}\end{aligned}$$

which means that

$$\frac{1}{2^n!}\left|\left\{\pi_n\in S_{2^n}:\left|y_n(j)-\frac{1}{2}\right|>t\right\}\right|<\frac{\mathrm{Var}(y_n(j))}{(t-|a_n|)^2}.$$

Thus, for any $A>0$ there are at most

$$\frac{2^n!2^n}{A}\cdot\frac{\mathrm{Var}(y_n(j))}{(t-|a_n|)^2}$$

index assignments $\pi_n\in S_{2^n}$, such that for each such π_n, there exist at least A codepoints $y_n(j)$ satisfying

$$\left|y_n(j)-\frac{1}{2}\right|>t.$$

Taking $A=\alpha 2^n$ we get the following equivalent conclusion. For any $\alpha\in(0,1)$, there are at most

$$\frac{2^n!}{\alpha}\cdot\frac{\mathrm{Var}(y_n(j))}{(t-|a_n|)^2}$$

index assignments $\pi_n\in S_{2^n}$, such that for each such π_n, there exist at least $\alpha 2^n$ codepoints $y_n(j)$ satisfying

$$\left|y_n(j)-\frac{1}{2}\right|>t.$$

This implies that for any $\alpha\in(0,1)$, there are at least

$$2^n!\left(1-\frac{\mathrm{Var}(y_n(j))}{\alpha(t-|a_n|)^2}\right)$$

index assignments $\pi_n\in S_{2^n}$ such that for each such π_n, there exist at most $\alpha 2^n$ codepoints $y_n(j)$ satisfying

$$\left|y_n(j)-\frac{1}{2}\right|>t.$$

A careful look at the variance shows a dependency on j but we can easily make a uniform upper bound on the variance which goes to zero at the speed $O(2^{-\beta n})$, where $\beta=-\log_2(1-2\epsilon+2\epsilon^2)>0$. We choose

$$t=\alpha=2^{-\beta n/4}.$$

This implies that for any n, a fraction of at least $1-O(2^{-\beta n/4})$ of all index assignments have the property that the fraction of codepoints $y_n(j)$ farther from $1/2$ than $2^{-\beta n/4}$, is at most $2^{-\beta n/4}$. In other words, as $n\to\infty$, an arbitrarily large fraction of all index assignments give rise to codebooks with an arbitrarily large fraction of codepoints arbitrarily close to $1/2$. □

Note that the proof of Theorem VI.4 demonstrates that the random mapping $\lambda_{\Pi_n}^{(n)}$ converges to zero in probability.

VII. Distortion Analysis

Let π_n be the index assignment for a rate-n quantizer with a uniform encoder on $[0,1]$ for a uniform source on $[0,1]$ and a binary symmetric channel with bit-error probability ϵ. Then the end-to-end MSE can be written as

$$\begin{aligned} D^{(\pi_n)} &= \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} p(\pi_n(j)|\pi_n(i)) \int_{i/2^n}^{(i+1)/2^n} (x-y_n(j))^2 dx \\ &= \frac{1}{3} + 2^{-n} \\ &\quad \cdot \sum_{i=0}^{2^n-1}\sum_{j=0}^{2^n-1} p(\pi_n(j)|\pi_n(i)) \left[y_n^2(j) - 2c_n(i)y_n(j)\right]. \end{aligned}$$

For any index assignment $\pi_n \in S_{2^n}$, let $D_{CU}^{(\pi_n)}$ denote the MSE of a decoder unoptimized uniform quantizer and let $D_{CO}^{(\pi_n)}$ denote the MSE of a decoder optimized uniform quantizer. For given ϵ and n, an index assignment $\pi_n \in S_{2^n}$ is said to be *optimal for a decoder unoptimized uniform quantizer* if for all $\pi'_n \in S_{2^n}$

$$D_{CU}^{(\pi_n)} \le D_{CU}^{(\pi'_n)}$$

and π_n is said to be *optimal for a decoder optimized uniform quantizer* if for all $\pi'_n \in S_{2^n}$

$$D_{CO}^{(\pi_n)} \le D_{CO}^{(\pi'_n)}.$$

Lemma VII.1: The MSE of a decoder optimized uniform quantizer with index assignment $\pi_n \in S_{2^n}$ is

$$D_{CO}^{(\pi_n)} = \frac{1}{3} - 2^{-n} \sum_{j=0}^{2^n-1} y_n^2(j).$$

Proof:

$$\begin{aligned} &D_{CO}^{(\pi_n)} \\ &\quad = \frac{1}{3} + 2^{-n} \\ &\quad\quad \cdot \sum_{j=0}^{2^n-1}\sum_{i=0}^{2^n-1} p(\pi_n(j)|\pi_n(i)) \left[y_n^2(j) - 2c_n(i)y_n(j)\right] \\ &\quad = \frac{1}{3} + 2^{-n} \\ &\quad\quad \cdot \sum_{j=0}^{2^n-1} \left[y_n^2(j) - 2y_n(j) \sum_{i=0}^{2^n-1} p(\pi_n(j)|\pi_n(i)) c_n(i) \right] \\ &\quad = \frac{1}{3} - 2^{-n} \sum_{j=0}^{2^n-1} y_n^2(j) \end{aligned} \tag{7.1}$$

where (7.1) follows from the weighted centroid condition. □

In [5], it was shown that randomly chosen index assignments for a decoder unoptimized uniform quantizer are asymptotically bad in the sense that their MSE approaches that of the worst possible index assignment in the limit as $n \to \infty$. The proof involved an explicit construction of a worst index assignment. The following theorem extends the result to a decoder optimized uniform quantizer and its proof does not require the construction of a worst case index assignment. In Theorem VII.2, the term $1/12$ is, in fact, the variance of the source.

Theorem VII.2: The MSE of a decoder optimized uniform quantizer is at most $1/12$, and for n sufficiently large, an arbitrarily large fraction of index assignments achieve an MSE arbitrarily close to $1/12$.

Proof: For any index assignment π_n, the average of the codepoints is

$$\begin{aligned} 2^{-n} \sum_{j=0}^{2^n-1} y_n(j) &= 2^{-n} \sum_{j=0}^{2^n-1}\sum_{i=0}^{2^n-1} c_n(i) p(\pi_n(j)|\pi_n(i)) \\ &= 2^{-n} \sum_{i=0}^{2^n-1} c_n(i) \\ &= \frac{1}{2}. \end{aligned}$$

Thus,

$$\begin{aligned} D_{CO}^{(\pi_n)} &= \frac{1}{3} - 2^{-n} \sum_{j=0}^{2^n-1} y_n^2(j) \\ &\le \frac{1}{3} - \left(2^{-n} \sum_{j=0}^{2^n-1} y_n(j) \right)^2 \qquad (7.2) \\ &= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \end{aligned}$$

where (7.2) follows from Jensen's inequality. The second assertion follows from Theorem VI.4 and Lemma VII.1. □

Although Theorem VII.2 indicates that asymptotically most index assignments yield MSEs close to $1/12$, in the following it will be shown that the NBC, the FBC, and the GC perform substantially better asymptotically.

The next two theorems give the MSEs for the NBC with a channel unoptimized decoder and with a channel optimized decoder. Theorem VII.3 was stated in [8] (see, e.g., [4] for a proof). The results are given as a function of the quantizer rate n and the channel bit-error probability ϵ. Analogous results are then given for the FBC, the GC, and the average for an index assignment chosen uniformly at random.

Theorem VII.3: The MSE of a decoder unoptimized uniform quantizer with the NBC index assignment is

$$D_{CU}^{(\text{NBC})} = \frac{2^{-2n}}{12} + \frac{\epsilon}{3}\left(1 - 2^{-2n}\right).$$

Theorem VII.4: The MSE of a decoder optimized uniform quantizer with the NBC index assignment is

$$D_{CO}^{(\text{NBC})} = \frac{2^{-2n}}{12} + \frac{\epsilon(1-\epsilon)}{3}\left(1 - 2^{-2n}\right).$$

Proof: Combining Proposition III.2 and Lemma VII.1 gives

$$\begin{aligned} D_{CO}^{(\text{NBC})} = \frac{1}{3} - 2^{-n} \sum_{j=0}^{2^n-1} \bigg[&\epsilon^2 + 2\epsilon 2^{-n}(1-2\epsilon)\left(j + \frac{1}{2}\right) \\ &+ 2^{-2n}(1-2\epsilon)^2 \left(j^2 + j + \frac{1}{4}\right) \bigg] \end{aligned}$$

$$= \frac{1}{3} - \left[\epsilon^2 + 2^{-n}\epsilon(1-2\epsilon) + 2^{-n}\epsilon(1-2\epsilon)(2^n-1) + 2^{-2n}(1-2\epsilon)^2 \cdot \left(\frac{(2^n-1)(2^{n+1}-1)}{6} + \frac{2^n-1}{2} + \frac{1}{4}\right)\right]$$
$$= \frac{2^{-2n}}{12} + \frac{\epsilon(1-\epsilon)}{3}\left(1-2^{-2n}\right). \quad \square$$

The next two theorems give the MSEs for the FBC with a channel unoptimized decoder and with a channel optimized decoder. Theorem VII.5 was given in [4]. The proof of Theorem VII.6 is similar to that of Theorem VII.4 and is omitted for brevity.

Theorem VII.5: The MSE of a decoder unoptimized uniform quantizer with the FBC index assignment is

$$D_{CU}^{(\mathrm{FBC})} = \frac{1}{12}\left(5\epsilon - 2\epsilon^2 + 2^{-2n}\left(1 - 8\epsilon + 8\epsilon^2\right)\right).$$

Theorem VII.6: The MSE of a decoder optimized uniform quantizer with the FBC index assignment is

$$D_{CO}^{(\mathrm{FBC})} = \frac{1}{12}\left(5\epsilon - 9\epsilon^2 + 8\epsilon^3 - 4\epsilon^4 - 2^{-2n}(1-2\epsilon)^4\right).$$

The next two theorems give the MSEs for the GC with a channel unoptimized decoder and with a channel optimized decoder. Theorem VII.7 was stated in [9] (see, e.g., [4] for a proof).

Theorem VII.7: The MSE of a decoder unoptimized uniform quantizer with the GC index assignment is

$$D_{CU}^{(\mathrm{GC})} = \frac{1}{6} - \frac{2^{-2n}}{12} - \frac{\left(\frac{1}{4} - \frac{\epsilon}{2}\right)\left(1 - \left(\frac{1}{4} - \frac{\epsilon}{2}\right)^n\right)}{\frac{3}{2} + \epsilon}.$$

Theorem VII.8: The MSE of a decoder optimized uniform quantizer with the GC index assignment is

$$D_{CO}^{(GC)} = \frac{1}{12} - \frac{1}{4} \cdot \frac{1 - \left(\frac{1}{2} - \epsilon\right)^{2n}}{\left(\frac{1}{2} - \epsilon\right)^{-2} - 1}.$$

Proof: Combining Proposition V.4 and Lemma VII.1 gives

$$D_{CO}^{(\mathrm{GC})} = \frac{1}{3} - 2^{-n} \cdot \sum_{j=0}^{2^n-1}\left[\frac{1}{4} + \frac{1}{2}\sum_{i=1}^{n}(-1)^{b_n(j,i)+1}\left(\frac{1}{2}-\epsilon\right)^i + \frac{1}{4}\sum_{i=1}^{n}\sum_{k=1}^{n}(-1)^{b_n(j,i)+b_n(j,k)+2}\left(\frac{1}{2}-\epsilon\right)^{i+k}\right]$$
$$= \frac{1}{12} - 2^{-n}\sum_{j=0}^{n}\left(y_n(j) - \frac{1}{2}\right) - 2^{-n-2} \cdot \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=0}^{2^n-1}(-1)^{b_n(j,i)+b_n(j,k)+2}\left(\frac{1}{2}-\epsilon\right)^{i+k}$$
$$= \frac{1}{12} - 2^{-n-2}\sum_{i=1}^{n}2^n\left(\frac{1}{2}-\epsilon\right)^{2i} \quad (7.3)$$
$$= \frac{1}{12} - \frac{\left(\frac{1}{2}-\epsilon\right)^2\left(1 - \left(\frac{1}{2}-\epsilon\right)^{2n}\right)}{4 - (1-2\epsilon)^2}$$

where (7.3) follows from the fact that the average of the codepoints for any index assignment is $1/2$ (see the proof of Theorem VII.2) and that for $i \neq k$, the sum $b_n(j,i) + b_n(j,k)$ is even 2^{n-1} times and odd 2^{n-1} times as j ranges between 0 and $2^n - 1$. $\square$

It can be seen from Theorems VII.3 and VII.4 that for the NBC, the reduction in MSE obtained by using a channel optimized quantizer decoder instead of one obeying the centroid condition, is $\epsilon^2(1-2^{-2n})/3$. For small ϵ, the MSE reduction is thus small. For a randomly chosen index assignment however, Theorems VII.9 and VII.10 show that channel optimized decoders reduce the average distortion by a factor of two over decoders obeying the centroid condition, independent of ϵ, in the limit as $n \to \infty$. Theorem VII.9 was stated in [8], and [5] contains a concise proof. Let $D_{CU}^{(\mathrm{RAN})}$ be a random variable denoting the MSE of a decoder unoptimized uniform quantizer with a randomly chosen index assignment.

Theorem VII.9: The average MSE of a decoder unoptimized uniform quantizer with an index assignment chosen uniformly at random is

$$E[D_{CU}^{(\mathrm{RAN})}] = \frac{2^{-2n}}{12} + \frac{1}{6} + \frac{1-(2^n+1)(1-\epsilon)^n}{6 \cdot 2^n}.$$

Since most index assignments are asymptotically bad, their average is bad as well. More precisely, the next theorem shows that the asymptotic average MSE of a decoder optimized uniform quantizer with an arbitrary index assignment converges to $1/12$, consistent with Theorem VII.2. Let $D_{CO}^{(\mathrm{RAN})}$ be a random variable denoting the MSE of a decoder optimized uniform quantizer with a randomly chosen index assignment.

Theorem VII.10: The average MSE of a decoder optimized uniform quantizer with an index assignment chosen uniformly at random is

$$E[D_{CO}^{(\mathrm{RAN})}] = \frac{2^{-2n}}{12} + \frac{1}{12} + \frac{1-(2^n+1)(1-2\epsilon+2\epsilon^2)^n}{12 \cdot 2^n}.$$

Proof: Let

$$a_n = \left(c_n(j) - \frac{1}{2}\right)\left(\frac{1}{1-2^{-n}}\right)\left((1-\epsilon)^n - 2^{-n}\right).$$

By Lemma VII.1, the expected value of $D_{CO}^{(\mathrm{RAN})}$ (over all index assignments) is

$$E[D_{CO}^{(\mathrm{RAN})}]$$
$$= \frac{1}{3} - 2^{-n}\sum_{j=0}^{2^n-1}E[y_n^2(j)]$$
$$= \frac{1}{3} - 2^{-n}\sum_{j=0}^{2^n-1}\left(\mathrm{Var}(y_n(j)) + E[y_n(j)]^2\right)$$
$$= \frac{1}{3} - 2^{-n}\sum_{j=0}^{2^n-1}\left(E\left[\left(y_n(j) - \frac{1}{2}\right)^2\right] + a_n + \frac{1}{4}\right) \quad (7.4)$$
$$= \frac{1}{12} - 2^{-n}\sum_{j=0}^{2^n-1}E\left[\left(y_n(j) - \frac{1}{2}\right)^2\right] \quad (7.5)$$
$$= \frac{1}{12} - 2^{-n}\sum_{j=0}^{2^n-1}\left[(1-\epsilon)^{2n}\left(c_n(j) - \frac{1}{2}\right)^2\right.$$

$$
\begin{aligned}
&+ (1-\epsilon)^{2n}\left(\frac{(1+\delta^2)^n - 1}{2^n - 1}\right)\\
&\cdot \left[\left(\sum_{i=0}^{2^n-1}\left(c_n(i)-\frac{1}{2}\right)^2\right) - \left(c_n(j)-\frac{1}{2}\right)^2\right]\\
&+ (1-\epsilon)^{2n}\\
&\cdot \left(\frac{2+(1+\delta)^{2n} - 2(1+\delta)^n - (1+\delta^2)^n}{(2^n-1)(2^n-2)}\right)\\
&\cdot \sum_{i\neq j}\sum_{l\neq i, l\neq j}\left(c_n(i)-\frac{1}{2}\right)\left(c_n(l)-\frac{1}{2}\right)\\
&+ (1-\epsilon)^{2n}\cdot 2\left(\frac{(1+\delta)^n - 1}{2^n-1}\right)\\
&\cdot \left.\sum_{l\neq j}\left(c_n(j)-\frac{1}{2}\right)\left(c_n(l)-\frac{1}{2}\right)\right] \qquad (7.6)
\end{aligned}
$$

$$
\begin{aligned}
= {} & \frac{1}{12} - 2^{-n}\\
&\cdot \sum_{j=0}^{2^n-1}\left[(1-\epsilon)^{2n}\left(c_n(j)-\frac{1}{2}\right)^2\right.\\
&+ (1-\epsilon)^{2n}\left(\frac{(1+\delta^2)^n-1}{2^n-1}\right)\\
&\cdot\left[\left(\frac{2^{2n}-1}{12\cdot 2^n}\right) - \left(c_n(j)-\frac{1}{2}\right)^2\right]\\
&+ (1-\epsilon)^{2n}\\
&\cdot \left(\frac{2+(1+\delta)^{2n} - 2(1+\delta)^n - (1+\delta^2)^n}{(2^n-1)(2^n-2)}\right)\\
&\cdot\left(2\left(c_n(j)-\frac{1}{2}\right)^2 - \frac{2^n}{12} + \frac{2^{-n}}{12}\right)\\
&\left. - 2(1-\epsilon)^{2n}\left(\frac{(1+\delta)^n-1}{2^n-1}\right)\left(c_n(j)-\frac{1}{2}\right)^2\right] \qquad (7.7)
\end{aligned}
$$

where (7.4) follows from Proposition VI.1 and (6.3), (7.5) follows from the fact that

$$\sum_{j=0}^{2^n-1}\left(c_n(j)-\frac{1}{2}\right) = 0$$

(7.6) follows from (6.6), and (7.7) results from the computations following (6.6). Passing the sum over j inside, distributing the factor of 2^{-n} over all terms, applying Lemma VI.2, and multiplying the $(1-\epsilon)^{2n}$ term through gives (7.8)–(7.9) at the bottom of the page, where (7.8) makes use of the computations following (6.6). □

Crimmins *et al.* [1] and McLaughlin, Neuhoff, and Ashley [3] showed that for every ϵ and every n the NBC is optimal for a decoder unoptimized uniform quantizer. We next extend the proof in [3] to show that for every ϵ and every n the NBC is also optimal for a decoder optimized uniform quantizer.

Lemma VII.11: Let Q_{π_n} denote the $2^n \times 2^n$ matrix whose (i,j)th elements are $q(\pi_n(i)|\pi_n(j))$. For any index assignment π_n, there exists a $2^n \times 2^n$ permutation matrix P such that

$$Q^2_{\pi_n} = PQ^2_{\pi_n^{(\mathrm{NBC})}}P^t.$$

Proof: Let P be the permutation matrix whose elements are

$$p_{i,j} = \begin{cases} 1 & \text{if } \pi_n(i) = j\\ 0 & \text{otherwise}\end{cases}$$

for $0 \le i,j \le 2^n-1$. Let $a_{i,j}$ and $b_{i,j}$, respectively, denote the (i,j)th elements of $Q_{\pi_n^{(\mathrm{NBC})}}$ and $PQ_{\pi_n^{(\mathrm{NBC})}}P^t$. Then

$$q(i|j) = a_{i,j} = b_{\pi_n^{-1}(i),\pi_n^{-1}(j)}$$

or equivalently

$$q(\pi_n(i)|\pi_n(j)) = a_{\pi_n(i),\pi_n(j)} = b_{i,j}$$

which implies

$$Q_{\pi_n} = PQ_{\pi_n^{(NBC)}}P^t.$$

Thus,

$$Q^2_{\pi_n} = PQ^2_{\pi_n^{(NBC)}}P^t$$

since P is orthogonal. □

Theorem VII.12: The NBC index assignment is optimal for a decoder optimized uniform quantizer, for every bit-error probability $\epsilon \ge 0$ and every quantizer rate $n \ge 1$.

Proof: Let

$$\underline{c} = [c_n(0), c_n(1), \ldots, c_n(2^n-1)]^t$$

and

$$\underline{y} = [y_n(0), y_n(1), \ldots, y_n(2^n-1)]^t$$

$$
\begin{aligned}
E[D_{CO}^{(RAN)}] = {} & \frac{1}{12} - \left\{\frac{(2^{2n}-1)(1-\epsilon)^{2n}}{12\cdot 2^{2n}} + \frac{(1-2\epsilon+2\epsilon^2)^n - (1-\epsilon)^{2n}}{2^n-1}\cdot\left(\frac{2^{2n}-1}{12\cdot 2^n} - \frac{2^{2n}-1}{12\cdot 2^{2n}}\right)\right.\\
&+ \frac{2(1-\epsilon)^{2n} + 1 - 2(1-\epsilon)^n - (1-2\epsilon+2\epsilon^2)^n}{(2^n-1)(2^n-2)}\cdot\left[2\left(\frac{2^{2n}-1}{12\cdot 2^{2n}}\right) - \left(\frac{2^{2n}-1}{12\cdot 2^n}\right)\right]\\
&\left. -2\left(\frac{(1-\epsilon)^n - (1-\epsilon)^{2n}}{2^n-1}\right)\cdot\left(\frac{2^{2n}-1}{12\cdot 2^{2n}}\right)\right\} \qquad (7.8)\\
= {} & \frac{1}{12} - \left[\frac{2^{2n}-1}{12\cdot 2^{2n}}\right]\cdot\left[\frac{2^n(1-2\epsilon+2\epsilon^2)^n - 1}{2^n-1}\right]\\
= {} & \frac{2^{-2n}}{12} + \frac{1}{12} + \frac{1-(2^n+1)(1-2\epsilon+2\epsilon^2)^n}{12\cdot 2^n}. \qquad (7.9)
\end{aligned}
$$

denote the column vectors of cell centroids and codepoints, respectively. Then Lemma VII.1, Lemma VII.11, and the weighted centroid condition imply that

$$\begin{aligned} D_{CO}^{(\pi_n)} &= \frac{1}{3} - 2^{-n}\|\underline{y}\|^2 \\ &= \frac{1}{3} - 2^{-n}\underline{c}^t Q_{\pi_n}^2 \underline{c} \\ &= \frac{1}{3} - 2^{-n}\underline{c}^t P Q_{\pi_n^{(NBC)}}^2 P^t \underline{c} \\ &= \frac{1}{3} - 2^{-n}\underline{z}^t Q_{\pi_n^{(NBC)}}^2 \underline{z} \\ &= \frac{1}{3} - 2^{-n}\underline{z}^t \hat{Q}_{\pi_n^{(NBC)}} \underline{z} \end{aligned} \quad (7.10)$$

where

$$\underline{z} = P^t \underline{c}$$

and where $\hat{Q}_{\pi_n^{(\mathrm{NBC})}}$ is the same as $Q_{\pi_n^{(\mathrm{NBC})}}$ but with ϵ replaced by $2\epsilon(1-\epsilon) \in (0, 1/2)$. McLaughlin, Neuhoff, and Ashley [3] showed that for every $\epsilon \in (0, 1/2)$, the quadratic form $\underline{z}^t Q_{\pi_n^{(\mathrm{NBC})}} \underline{z}$ (and thus in particular $\underline{z}^t \hat{Q}_{\pi_n^{(\mathrm{NBC})}} \underline{z}$) is maximized for uniform sources and uniform quantizers satisfying $\sum_i c_n(i) = 0$, when $\pi_n = \pi_n^{(\mathrm{NBC})}$. Shifting the support of a uniform source from $[0, 1]$ to $[-1/2, 1/2]$ changes each term in (7.10) by a constant term, independent of the index assignment. Thus $D_{CO}^{(\pi_n)}$ is minimized when $\pi_n = \pi_n^{(\mathrm{NBC})}$, and therefore the NBC is optimal for decoder optimized uniform quantizers for all ϵ and n. □

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