Upper Bounds for Constant-Weight Codes

Erik Agrell, *Member, IEEE*, Alexander Vardy, *Fellow, IEEE*, and Kenneth Zeger, *Fellow, IEEE*

***Abstract*—Let $A(n, d, w)$ denote the maximum possible number of codewords in an (n, d, w) constant-weight binary code. We improve upon the best known upper bounds on $A(n, d, w)$ in numerous instances for $n \leqslant 24$ and $d \leqslant 12$, which is the parameter range of existing tables. Most improvements occur for $d = 8, 10$, where we reduce the upper bounds in more than half of the unresolved cases. We also extend the existing tables up to $n \leqslant 28$ and $d \leqslant 14$.**

To obtain these results, we develop new techniques and introduce new classes of codes. We derive a number of general bounds on $A(n, d, w)$ by means of mapping constant-weight codes into Euclidean space. This approach produces, among other results, a bound on $A(n, d, w)$ that is tighter than the Johnson bound. A similar improvement over the best known bounds for doubly-constant-weight codes, studied by Johnson and Levenshtein, is obtained in the same way. Furthermore, we introduce the concept of doubly-bounded-weight codes, which may be thought of as a generalization of the doubly-constant-weight codes. Subsequently, a class of Euclidean-space codes, called zonal codes, is introduced, and a bound on the size of such codes is established. This is used to derive bounds for doubly-bounded-weight codes, which are in turn used to derive bounds on $A(n, d, w)$. We also develop a universal method to establish constraints that augment the Delsarte inequalities for constant-weight codes, used in the linear programming bound.

In addition, we present a detailed survey of known upper bounds for constant-weight codes, and sharpen these bounds in several cases. All these bounds, along with all known dependencies among them, are then combined in a coherent framework that is amenable to analysis by computer. This improves the bounds on $A(n, d, w)$ even further for a large number of instances of n, d, and w.

***Index Terms*—Constant-weight codes, Delsarte inequalities, doubly-bounded-weight codes, doubly-constant-weight codes, error-correcting codes, linear programming, spherical codes, zonal codes**
.

I. Introduction

An (n, d, w) constant-weight binary code is a set of binary vectors of length n, such that each vector contains w ones and $n - w$ zeros, and any two vectors differ in at least d positions. Given the three parameters: length n, weight w, and distance d, what is the largest possible size $A(n, d, w)$ of an (n, d, w) constant-weight binary code? This question has been studied for almost four decades, and remains one of the most basic questions in coding theory.

Although the general answer is not known, various upper and lower bounds on $A(n, d, w)$ have been developed. Lower bounds are typically obtained by means of explicit code constructions, while upper bounds involve analytic methods, ranging from linear programming to geometry.

The first systematic tables of bounds on $A(n, d, w)$ appeared in 1977 in the book of MacWilliams and Sloane [42, pp. 684–691], for $n \leqslant 24$ and $d \leqslant 10$. An updated version of these tables, along with a more complete treatment of the underlying theory, was published [8] in 1978. Another update appeared in Honkala's Licentiate thesis [34, Sec. 6], together with a new table of upper bounds for $d = 12$ and $n \leqslant 27$. Since then, there has been very little progress on the upper bounds. In contrast, lower bounds on $A(n, d, w)$ were improved upon many times. The lower bounds of [8] were revised in 1980 by Graham and Sloane [31]. Then in 1990, following a large number of new explicit code constructions for certain parameters, came the encyclopedic work of Brouwer, Shearer, Sloane, and Smith [17], where the best known lower bounds on $A(n, d, w)$ for $n \leqslant 28$ and $d \leqslant 18$ are collected. Upper bounds are given in [17] only for those parameters where these bounds are known to coincide with the lower bounds.

This work is concerned with the problem of determining upper bounds on the size of constant-weight codes. Our contributions to this problem are three-fold, as described in the next three paragraphs.

First, we improve upon the existing upper bounds on $A(n, d, w)$ in many instances. For example, out of the 23 unresolved cases for $d = 8$ in [17], [34], fourteen upper bounds are improved upon in this paper. For $d = 10$, we update 10 out of the 18 unresolved cases. As a result, we establish seven new exact values of $A(n, d, w)$, and rederive by analytical methods exact values of $A(n, d, w)$ that were previously found by exhaustive computer search. Furthermore, we extend the existing tables of upper bounds on $A(n, d, w)$ from $n \leqslant 24$ and $d \leqslant 10$ to $n \leqslant 28$ and $d \leqslant 14$, so as to match the tables of lower bounds in [17]. In fact, our intent in the present paper is to provide a counterpart to [17], with respect to the upper bounds on $A(n, d, w)$.

Second, in addition to the specific bounds on $A(n, d, w)$ mentioned in the foregoing paragraph, we develop a number of new general approaches to the problem. Some of these are briefly described below. It is well known, since the work of Johnson [35] and Levenshtein [39], that certain bounds on $A(n, d, w)$ can be derived using doubly-constant-weight

Manuscript received December 14, 1999; revised June 29, 2000. This work was supported in part by the National Science Foundation, David and Lucile Packard Foundation, Stiftelsen ISS'90, and Svensk Informations- och Mikrografiorganisation. This work was carried out in part while E. Agrell was visiting the University of California, San Diego. The material in this paper was presented in part at the IEEE International Symposium on Information Theory, Sorrento, Italy, June 2000.

E. Agrell is with the Department of Electrical and Computer Engineering, Chalmers Lindholmen University College, 40272 Göteborg, Sweden (e-mail: agrell@chl.chalmers.se).

A. Vardy and K. Zeger are with the Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, CA 92093-0407 USA (e-mail: vardy@montblanc.ucsd.edu; zeger@ucsd.edu).

Communicated by A. Barg, Associate Editor for Coding Theory.

Publisher Item Identifier S 0018-9448(00)09860-6.

codes, which constitute a special restricted subclass of constant-weight codes. In this work, we introduce the concept of doubly-bounded-weight codes. These codes are less restricted than doubly-constant-weight codes, yet more restricted than general constant-weight codes. We derive bounds on the size of doubly-bounded-weight codes, which turn out to be extremely useful in developing upper bounds on $A(n, d, w)$. Another useful approach, developed in Section III of this paper, is as follows. Map the three types of constant-weight codes into Euclidean space. It is shown in Section III that, under the appropriate mapping, this results in three different kinds of spherical codes. Consequently, one can use upper bounds for spherical codes (already known bounds, as well as new bounds for zonal codes derived in Appendix B) to establish bounds on constant-weight codes. Surprisingly, this simple idea often leads to powerful upper bounds on $A(n, d, w)$ (cf. Examples 2–4). Finally, as in most previous work on the subject, we make use of linear programming, based on the Delsarte [24] inequalities for constant-weight codes. It is known that the distance distribution of constant-weight codes is subject to more constraints than can be obtained from the Delsarte inequalities, but determining these extra constraints has in most cases involved a different (nontrivial) manipulation for each distinct set of parameters (n, d, w). In contrast, in this work, we develop a universal method to find such constraints (cf. Proposition 17).

Our third contribution is the integration of all the known (to us) bounds on constant-weight codes—as well as related methods and techniques—into a coherent framework that is amenable to analysis by computer. Many existing bounds on $A(n, d, w)$ are restated herein in a different, substantially simplified, way. Other known bounds whose application was previously limited to specific sets of parameters (n, d, w) are given here in their most general form. We list *all* methods that we are aware of to obtain upper bounds on $A(n, d, w)$. The methods are of two types: dependent and stand-alone. Dependent bounds are functions of other bounds, whereas stand-alone bounds are not. Most of the known bounds are dependent, which makes their evaluation, and the determination of which bound is best for a given set of parameters, a fairly complex process. These dependencies are outlined in Fig. 1, where each arrowhead represents one bound, as given by a numbered theorem in this paper. (We have omitted the stand-alone bounds in Fig. 1.) Thus several steps may be necessary to prove a tight bound on $A(n, d, w)$ for specific n, d, and w. The organization of all these methods into a streamlined framework has the advantage that the paths in Fig. 1 can be followed iteratively until a steady state is reached. Later in this paper, we give a series of examples that will illustrate one such route in Fig. 1.

Since the early work of Johnson [35] and Freiman [30], bounds on constant-weight codes have been employed to derive bounds on unrestricted binary codes. An (n, d) binary code (unrestricted) is a set of binary vectors of length n such that any two of them differ in at least d positions; the maximum number of codewords in any such code is usually denoted $A(n, d)$. An important relation between $A(n, d)$ and $A(n, d, w)$ is due to Elias (see [10, pp. 451, 456]) and Bassalygo [6]. This elegant

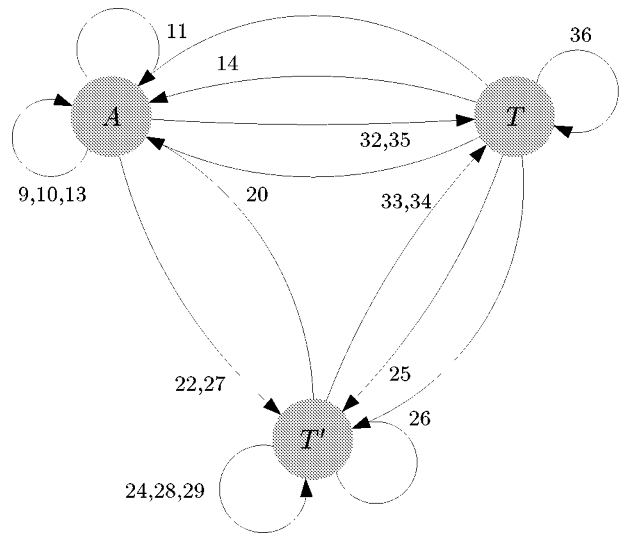

Fig. 1. The interdependence between bounds on the three types of binary constant-weight codes: A stands for general constant-weight codes, T' for doubly-bounded-weight codes, and T for doubly-constant-weight codes. Numbers refer to theorems in this paper. For example, the arrowhead labeled 20 represents a bound on $A(n, d, w)$, derived in Theorem 20, in terms of bounds on doubly-constant-weight codes and bounds on doubly-bounded-weight codes.

Bassalygo–Elias inequality

$$A(n,d) \leqslant \frac{2^n}{\binom{n}{w}} A(n, d, w) \tag{1}$$

was improved upon by Levenshtein [39, eq. (32)], and later by van Pul (see [1]), who pointed out that the right-hand side of (1) can be reduced by a factor of two. The best known asymptotic upper bound on $A(n, d)$, given by McEliece, Rodemich, Rumsey, and Welch [43] in 1977, consists of this inequality in conjunction with a linear programming bound on the size of constant-weight codes. Thus it should not be surprising that better bounds on $A(n, d, w)$ lead to new bounds on $A(n, d)$. Our contributions in the area of unrestricted codes, based on the results of this paper, will be presented elsewhere.

While unrestricted codes have obvious applications in error correction, constant-weight codes have been historically regarded as a purely theoretical construction. Today, however, they are generally recognized as an important class of codes in their own right. They have been recently introduced in a number of engineering applications, including code-division multiple-access (CDMA) systems for optical fibers [19], protocol design for the collision channel without feedback [1], automatic-repeat-request error-control systems [54], and parallel asynchronous communication [12]. In addition, they often serve as building blocks in the design of spherical codes [28] and DC-free constrained codes [29], [52]. Further applications have been reported in frequency-hopping spread-spectrum systems, radar and sonar signal design, mobile radio, and synchronization [9], [11], [19]. For general background on constant-weight codes, and the related class of spherical codes, we refer the reader to [22], [28], and [42].

The rest of this paper is organized as follows. In the next section, we define concepts and terminology that will be used throughout this work. A simple mapping from binary codes to spherical codes is introduced in Section III; bounds derived directly from this mapping improve upon two well-known bounds by Johnson. Sections IV–VI list all useful upper bounds on constant-weight codes that we are aware of, including many new ones derived in this paper. One section is devoted to each of the three classes: constant-weight codes, doubly-bounded-weight codes, and doubly-constant-weight codes. Finally, tables of the best known upper bounds on $A(n, d, w)$ are presented in Section VII, for all $n \leqslant 28$.

II. Preliminaries

In this section, we introduce concepts and notation that will be used throughout the paper. We distinguish between codes in Hamming space (that is, binary codes) and their counterparts in Euclidean space—the spherical codes.

A. Hamming Space

Four nested levels of binary codes will be discussed. To begin with, any subset of $\mathcal{H}(n) = \{0, 1\}^n$ is called an *unrestricted binary code*, in the sense that no weight constraint is imposed. A *constant-weight binary code* is any subset of

$$\mathcal{H}(n, w) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathcal{H}(n): \boldsymbol{x} \cdot \mathbf{1} = w\} \tag{2}$$

where $\mathbf{1}$ is the all-one vector and the dot product is carried out in $\mathbb{R}^n$. A *doubly-bounded-weight code* is a constant-weight code with at most w_1 ones in the first n_1 positions and at least w_2 ones in the last n_2 positions. (In the following, the first n_1 positions will be called the *head* and the last n_2 positions the *tail*.) Equivalently, a doubly-bounded-weight code is a subset of

$$\mathcal{H}'(w_1, n_1, w_2, n_2) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathcal{H}(n_1 + n_2, w_1 + w_2): \boldsymbol{x} \cdot \boldsymbol{u}_1 \leqslant w_1\} \tag{3}$$

where

$$\boldsymbol{u}_1 \stackrel{\text{def}}{=} (\overbrace{\underbrace{1, \ldots, 1}_{n_1}}^{\text{head}}, \overbrace{\underbrace{0, \ldots, 0}_{n_2}}^{\text{tail}}). \tag{4}$$

Finally, a *doubly-constant-weight code* is any subset of

$$\mathcal{H}(w_1, n_1, w_2, n_2) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathcal{H}(n_1 + n_2, w_1 + w_2): \boldsymbol{x} \cdot \boldsymbol{u}_1 = w_1\}. \tag{5}$$

Thus a codeword of a doubly-constant-weight codeword has exactly w_1 ones in its head and w_2 ones in its tail. It follows directly from the definitions in (2), (3), and (5) that doubly-constant-weight codes constitute a subclass of the doubly-bounded-weight codes, which themselves constitute a subclass of the constant-weight codes, which, in turn, are a subclass of unrestricted codes.

Unrestricted codes and constant-weight codes have been studied extensively in the past. Doubly-constant-weight codes were proposed in [39] and [37]. The class of doubly-bounded-weight codes is introduced in this paper; it turns out to be very useful in deriving bounds for the other classes.

In the following, $d(\mathcal{C})$ denotes the minimum Hamming distance within a code $\mathcal{C}$, namely,

$$d(\mathcal{C}) \stackrel{\text{def}}{=} \min_{\substack{\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathcal{C} \\ \boldsymbol{c}_1 \neq \boldsymbol{c}_2}} d(\boldsymbol{c}_1, \boldsymbol{c}_2) \tag{6}$$

where $d(\boldsymbol{c}_1, \boldsymbol{c}_2)$ is the number of positions in which the codewords $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ differ. Given a set $\mathcal{U} \subseteq \mathcal{H}(n)$, let

$$\Phi(\mathcal{U}, d) \stackrel{\text{def}}{=} \{\mathcal{C} \subseteq \mathcal{U}: d(\mathcal{C}) \geqslant d\} \tag{7}$$

denote all subsets of $\mathcal{U}$ whose minimum distance is at least d. We are interested in the quantities

$$A(n, d) \stackrel{\text{def}}{=} \max_{\mathcal{C} \in \Phi(\mathcal{H}(n), d)} |\mathcal{C}| \tag{8}$$

$$A(n, d, w) \stackrel{\text{def}}{=} \max_{\mathcal{C} \in \Phi(\mathcal{H}(n, w), d)} |\mathcal{C}| \tag{9}$$

where $0 \leqslant d \leqslant n$ and $0 \leqslant w \leqslant n$, as well as

$$T'(w_1, n_1, w_2, n_2, d) \stackrel{\text{def}}{=} \max_{\mathcal{C} \in \Phi(\mathcal{H}'(w_1, n_1, w_2, n_2), d)} |\mathcal{C}|$$

$$T(w_1, n_1, w_2, n_2, d) \stackrel{\text{def}}{=} \max_{\mathcal{C} \in \Phi(\mathcal{H}(w_1, n_1, w_2, n_2), d)} |\mathcal{C}|$$

where $0 \leqslant w_1 \leqslant n_1$, $0 \leqslant w_2 \leqslant n_2$, and $0 \leqslant d \leqslant n_1 + n_2$. Despite the potential confusion of using $A(\cdot)$ for both (8) and (9), we maintain this standard notation [17], [42].

B. Euclidean Space

We start by defining, in analogy to (6) and (7), the distance and the Φ functions in Euclidean space, as follows:

$$d_E(\mathcal{C}) \stackrel{\text{def}}{=} \min_{\substack{\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathcal{C} \\ \boldsymbol{c}_1 \neq \boldsymbol{c}_2}} \|\boldsymbol{c}_1 - \boldsymbol{c}_2\|$$

$$\Phi_E(\mathcal{U}, d_E) \stackrel{\text{def}}{=} \{\mathcal{C} \subseteq \mathcal{U}: d_E(\mathcal{C}) \geqslant d_E\}.$$

Here $\|\cdot\|$ is the Euclidean norm, $\mathcal{C}$ is a finite subset of $\mathbb{R}^n$, and $\mathcal{U}$ is an arbitrary subset of $\mathbb{R}^n$.

Two types of codes in Euclidean space will be considered. The *unit sphere* is the set

$$\mathcal{S}(n) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathbb{R}^n: \|\boldsymbol{x}\| = 1\}.$$

A *spherical code* is a finite subset of $\mathcal{S}(n)$. To characterize the codeword separation in a spherical code, the *minimum angle* ϕ or the *maximum cosine* s is often used instead of the Euclidean distance. The relation between these three parameters is

$$s \stackrel{\text{def}}{=} \cos\phi = 1 - \frac{d_E^2}{2}. \tag{10}$$

We will generally use s as the separation parameter. The maximum possible cardinality of an n-dimensional spherical code with maximum cosine s is

$$A_S(n, s) \stackrel{\text{def}}{=} \max_{\mathcal{C} \in \Phi_E(\mathcal{S}(n), \sqrt{2-2s})} |\mathcal{C}|.$$

For $s > 0$, the best known general upper bound on $A_S(n, s)$ was given by Levenshtein in [40]. This bound can be improved upon for certain specific parameters using the methods of Boyvalenkov, Danev, and Bumova [15].

For $s \leqslant 0$, this function is known exactly. Specifically, it is known that

$$A_S(n, s) = \left\lfloor 1 - \frac{1}{s} \right\rfloor, \qquad \text{if } s \leqslant -\frac{1}{n} \tag{11}$$

$$A_S(n, s) = n + 1, \qquad \text{if } -\frac{1}{n} \leqslant s < 0 \tag{12}$$

$$A_S(n, 0) = 2n. \tag{13}$$

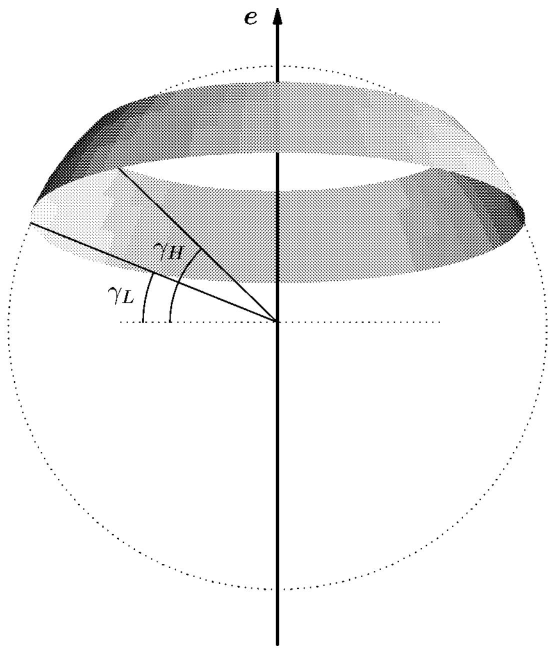

Fig. 2. A zone.

Rankin [47] was the first to establish (11), while (12) was originally stated by Davenport and Hajós [23], and proved by Aczél and Szele [2]. Equation (13) was first stated by Erdös [26], and proved by Sarkadi and Szele [50].

Example 1: We have $A_S(25, -3/41) = \lfloor 44/3 \rfloor = 14$ (to be continued in Example 15). □

We now introduce the class of zonal codes. A *zone* is a subset of a sphere bounded by two parallel hyperplanes [56, pp. 314–315], as illustrated in Fig. 2. Given a "north pole" vector $\boldsymbol{e}$, with $\|\boldsymbol{e}\| = 1$, we define

$$\mathcal{Z}(n, \gamma_L, \gamma_H, \boldsymbol{e}) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathcal{S}(n) \colon \sin\gamma_L \leqslant \boldsymbol{x} \cdot \boldsymbol{e} \leqslant \sin\gamma_H\}$$

where $-\pi/2 \leqslant \gamma_L \leqslant \gamma_H \leqslant \pi/2$. A zone with $\gamma_H = \pi/2$ is a spherical cap [56, pp. 314–315]. A *zonal code* is a finite subset of a zone. The maximum cardinality of a zonal code is denoted

$$A_Z(n, s, \gamma_L, \gamma_H) \stackrel{\text{def}}{=} \max |\mathcal{C}| \tag{14}$$

where the maximum is taken over all

$$\mathcal{C} \in \Phi_E\left(\mathcal{Z}(n, \gamma_L, \gamma_H, \boldsymbol{e}), \sqrt{2 - 2s}\right).$$

Clearly, the right-hand side of (14) is independent of $\boldsymbol{e}$.

III. Bounds from Spherical Codes

It is well known that, under a suitable mapping, the class of binary codes can be viewed as a subclass of spherical codes. This implies that a lower bound on the size of binary codes is also a lower bound for spherical codes. Conversely, an upper bound on the cardinality of spherical codes serves as an upper bound for binary codes. The former relation has been successfully exploited—see [22, pp. 26–27], [27], [28], and references therein. One contribution of the present paper is to investigate the latter relation, from which we obtain improved bounds in some cases.

This approach, which has been less highlighted than its converse, was used in [27] to prove two well-known bounds; see below in Section III-B. A somewhat related method was suggested by Wax [55], who derived upper bounds[1] on binary codes from some sphere packings (not spherical codes) in Euclidean space.

A. Binary Codes as Spherical Codes

We first map three of the classes of binary codes introduced in the previous section into Euclidean space. This mapping produces spherical codes in three different dimensions. Known upper bounds for spherical codes are then used to generate new upper bounds for the original binary codes. The derivation of an analogous bound for doubly-bounded-weight codes is deferred to Section V-B.

Let $\Omega(\cdot)$ denote the mapping $0 \to 1$ and $1 \to -1$ from binary Hamming space to Euclidean space. Then

$$\Omega(\mathcal{H}(n)) = \{1, -1\}^n \tag{15}$$

$$\Omega(\mathcal{H}(n, w)) = \{\boldsymbol{x} \in \Omega(\mathcal{H}(n)) \colon \boldsymbol{x} \cdot \mathbf{1} = n - 2w\} \tag{16}$$

$$\Omega(\mathcal{H}'(w_1, n_1, w_2, n_2)) = \{\boldsymbol{x} \in \Omega(\mathcal{H}(n, w)) \colon \boldsymbol{x} \cdot \boldsymbol{u}_1 \geqslant n_1 - 2w_1\} \tag{17}$$

$$\Omega(\mathcal{H}(w_1, n_1, w_2, n_2)) = \{\boldsymbol{x} \in \Omega(\mathcal{H}(n, w)) \colon \boldsymbol{x} \cdot \boldsymbol{u}_1 = n_1 - 2w_1\} \tag{18}$$

where $n = n_1 + n_2$, $w = w_1 + w_2$, and $\boldsymbol{u}_1$ is as defined in (4). Note that if the Hamming distance between two binary vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is d, then the Euclidean distance between $\Omega(\boldsymbol{x}_1)$ and $\Omega(\boldsymbol{x}_2)$ is $2\sqrt{d}$.

Clearly, $\Omega(\mathcal{H}(n))$ is a subset of the n-dimensional hypersphere of radius $r_0 = \sqrt{n}$, centered at $\boldsymbol{c}_0 = \mathbf{0}$.

For constant-weight codes, any point $\boldsymbol{x} \in \mathcal{H}(n, w)$ satisfies $(\Omega(\boldsymbol{x}) - \boldsymbol{c}_1) \cdot \mathbf{1} = 0$ and $\|\Omega(\boldsymbol{x}) - \boldsymbol{c}_1\| = r_1$, where

$$r_1 = 2\sqrt{\frac{w(n-w)}{n}} \tag{19}$$

and

$$\boldsymbol{c}_1 = \left(1 - \frac{2w}{n}, \ldots, 1 - \frac{2w}{n}\right). \tag{20}$$

Hence $\Omega(\mathcal{H}(n, w))$ is a subset of the $(n-1)$-dimensional hypersphere of radius r_1 centered at $\boldsymbol{c}_1$.

In a similar way, one can show that $\Omega(\mathcal{H}(w_1, n_1, w_2, n_2))$ is a subset of the $(n_1 + n_2 - 2)$-dimensional hypersphere of radius

$$r_2 = 2\sqrt{\frac{w_1(n_1 - w_1)}{n_1} + \frac{w_2(n_2 - w_2)}{n_2}}$$

centered at

$$\boldsymbol{c}_2 = \left(1 - \frac{2w_1}{n_1}\right)\boldsymbol{u}_1 + \left(1 - \frac{2w_2}{n_2}\right)\boldsymbol{u}_2$$

where $\boldsymbol{u}_1$ is as defined in (4) and $\boldsymbol{u}_2 = \mathbf{1} - \boldsymbol{u}_1$. This follows from the fact that for any point $\boldsymbol{x} \in \mathcal{H}(w_1, n_1, w_2, n_2)$, we have $(\Omega(\boldsymbol{x}) - \boldsymbol{c}_2) \cdot \boldsymbol{u}_1 = (\Omega(\boldsymbol{x}) - \boldsymbol{c}_2) \cdot \boldsymbol{u}_2 = 0$ and $\|\Omega(\boldsymbol{x}) - \boldsymbol{c}_2\| = r_2$.

These observations lead to upper bounds on the size of the corresponding binary codes, formulated in terms of the maximum cardinality of spherical codes.

Theorem 1:

$$A(n, 2\delta) \leqslant A_S(n, s)$$

[1]These bounds are not very strong, however. See Appendix A.

where

$$s = 1 - 4\frac{\delta}{n}.$$

Theorem 2:

$$A(n, 2\delta, w) \leqslant A_S(n-1, s), \qquad \text{if } s \geqslant -1 \tag{21}$$
$$A(n, 2\delta, w) = 1, \qquad \text{if } s < -1 \tag{22}$$

where

$$s = 1 - \frac{\delta n}{w(n-w)}.$$

Proof: Let $\mathcal{C}$ be a constant-weight code with parameters $(n,\, 2\delta,\, w)$. Translating $\Omega(\mathcal{C})$ by $-\boldsymbol{c}_1$ and scaling the result by $1/r_1$, in accordance with (20) and (19), yields an $(n-1)$-dimensional spherical code. Its maximum cosine is given by (10), where $d_E = (2/r_1)\sqrt{2\delta}$. Using $A_S(n-1,\, s)$ as an upper bound for $|\Omega(\mathcal{C})|$ completes the proof. ∎

Theorem 3:

$$T(w_1, n_1, w_2, n_2, 2\delta) \leqslant A_S(n_1 + n_2 - 2, s), \qquad \text{if } s \geqslant -1$$
$$T(w_1, n_1, w_2, n_2, 2\delta) = 1, \qquad \text{if } s < -1$$

where

$$s = 1 - \frac{\delta n_1 n_2}{n_1 w_2 (n_2 - w_2) + n_2 w_1 (n_1 - w_1)}.$$

The proofs of all three theorems are similar to each other, and their common principle is demonstrated in the proof of Theorem 2.

Note that the case $s < -1$ corresponds to a spherical code whose minimum Euclidean distance is greater than the diameter of the sphere. Although formally $A_S(n,\, s) = 1$ for such s, we chose to treat this trivial case separately.

B. New Bounds

For $s \leqslant 0$, the exact values of $A_S(n,\, s)$ given by (11) and (13) can be used in conjunction with Theorems 1–3 to yield bounds on the size of binary codes. The method is simple and produces interesting results.

The resulting bounds, which are summarized in the following three corollaries, can be interpreted as a common framework for bounds by Plotkin, Johnson, and Levenshtein, as well as some new, tighter, bounds. The bounds (23) and (25) were derived in [27] using this method.

Corollary 4:

$$A(n, 2\delta) \leqslant \left\lfloor \frac{4\delta}{4\delta - n} \right\rfloor, \qquad \text{if } 4\delta > n \tag{23}$$
$$A(n, 2\delta) \leqslant 2n, \qquad \text{if } 4\delta = n. \tag{24}$$

Corollary 5:

$$A(n, 2\delta, w) \leqslant \left\lfloor \frac{\delta}{b} \right\rfloor, \qquad \text{if } b \geqslant \frac{\delta}{n} \tag{25}$$
$$A(n, 2\delta, w) \leqslant n, \qquad \text{if } 0 < b \leqslant \frac{\delta}{n} \tag{26}$$
$$A(n, 2\delta, w) \leqslant 2n - 2, \qquad \text{if } b = 0 \tag{27}$$

where

$$b = \delta - \frac{w(n-w)}{n}.$$

Corollary 6:

$$T(w_1, n_1, w_2, n_2, 2\delta) \leqslant \left\lfloor \frac{\delta}{b} \right\rfloor, \qquad \text{if } b \geqslant \frac{\delta}{n_1 + n_2 - 1} \tag{28}$$
$$T(w_1, n_1, w_2, n_2, 2\delta) \leqslant n_1 + n_2 - 1, \qquad \text{if } 0 < b \leqslant \frac{\delta}{n_1 + n_2 - 1} \tag{29}$$
$$T(w_1, n_1, w_2, n_2, 2\delta) \leqslant 2n_1 + 2n_2 - 4, \qquad \text{if } b = 0 \tag{30}$$

where

$$b = \delta - \frac{w_1(n_1 - w_1)}{n_1} - \frac{w_2(n_2 - w_2)}{n_2}.$$

Corollary 4 is similar to the Plotkin bound [44]. The only difference is that in the latter, the right-hand side of (23) is truncated to an even value, instead of just an integer as in Corollary 4. Hence the Plotkin bound is stronger. It was derived using an entirely different (combinatorial) technique, as will be mentioned in the context of Proposition 7.

For $b > \delta/(n+1)$, Corollary 5 is equivalent to one of Johnson's bounds [35]. Johnson showed (25) for all $b > 0$ by the same method that is used below to prove Theorem 29. If we let $\delta = w$, Corollary 5 yields

$$A(n, 2w, w) \leqslant \left\lfloor \frac{n}{w} \right\rfloor \tag{31}$$

which is another well-known special case [39], [42, p. 525]. Note also that (22) is covered by (25). The bound (26), which improves on the Johnson bound for $0 \leqslant b \leqslant \delta/(n+1)$, has not, to our knowledge, been previously published. Comparing Corollary 5 with Levenshtein's linear programming bound [41, Theorem 6.25], it can be observed that (25) is equivalent to Levenshtein's bound within the applicable range of parameters, (26) is lower, and (27) is higher. Hence, (27) need not be further considered.

The inequalities (29) and (30) in Corollary 6 appear to be new, whereas (28) was found previously by both Levenshtein [39] and Johnson [37]. They use this inequality for all $b > 0$ (see also Section V-A).

Example 2: Take $(n,\, 2\delta,\, w) = (24,\, 10,\, 7)$. Corollary 5 gives $b = 1/24$ and $A(24,\, 10,\, 7) \leqslant 24$. This is an improvement on the best previously known upper bound of 27, given in [31]. Since a lower bound of 24 is known [17], we conclude that this bound is in fact tight. □

Example 3: Corollary 5 also gives $A(12,\, 6,\, 5) \leqslant 12$. This reproduces a well-known bound which was proved in [36] through a combinatorial argument specifically devised for these parameters. See also [42, p. 530]. □

Example 4: For $(w_1,\, n_1,\, w_2,\, n_2,\, 2\delta) = (4,\, 9,\, 4,\, 13,\, 10)$, Corollary 6 yields $b = 1/117$ and $T(4,\, 9,\, 4,\, 13,\, 10) \leqslant 21$, a significant improvement upon the best previously known bound of 29, given in [8]. For $T(2, 9, 6, 14, 10)$, Corollary 6 reduces the best known upper bound from 30 to 22. □

C. Plotkin-Type Bounds

It is somewhat surprising that Corollaries 4–6 are so similar to the Plotkin bound and its various relatives, since these bounds have been derived using entirely different methods. For comparison and for future reference, we now re-establish the Plotkin bound in its most general form following the traditional, combinatorial, approach. From this generic form of the Plotkin bound, many related bounds easily follow. Special cases include the original Plotkin bound, four of Johnson's and Levenshtein's bounds, as well as a new bound to be reported in Section V-A.

Given a code $\mathcal{C} \subseteq \mathcal{H}(n)$, let f_i denote the proportion of codewords that have a one in position i. We have the following proposition.

Proposition 7: Let $\mathcal{C} \in \Phi(\mathcal{H}(n), 2\delta)$. Then

$$|\mathcal{C}| \leqslant \frac{\delta}{\delta - \sum_{i=1}^{n} f_i(1-f_i)} \tag{32}$$

provided that the denominator is positive.

Proof: We consider the average distance within the code $\mathcal{C}$, defined as follows:

$$d_{\text{av}} \stackrel{\text{def}}{=} \frac{1}{M(M-1)} \sum_{\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathcal{C}} d(\boldsymbol{c}_1, \boldsymbol{c}_2) \tag{33}$$

where $M = |\mathcal{C}|$. For each $\boldsymbol{c} \in \mathcal{C}$, count the contribution to the sum on the right-hand side of (33) from each position. Then, interchanging the order of summation, it is easy to see that

$$d_{\text{av}} = \frac{2M}{M-1} \sum_{i=1}^{n} f_i(1-f_i).$$

The proposition now follows from the fact that $d_{\text{av}} \geqslant 2\delta$. ■

Bounds for many types of binary codes can be derived from Proposition 7, since constraints on codewords translate into constraints on $f_1, \ldots, f_n$. For instance, using no information other than $0 \leqslant f_i \leqslant 1$ for all i, we find that the maximum of $\sum f_i(1-f_i)$ is $n/4$. Substituting $n/4$ for the sum in (32) establishes (23). If, in addition, $f_1, \ldots, f_n$ are constrained to be multiples of $1/M$, the resulting bound is the classical Plotkin bound of [44].

Bounds for constant-weight codes are obtained from Proposition 7 by requiring $f_1 + \cdots + f_n = w$. If this is the only constraint in the maximization, the result is a proof of the aforementioned Johnson bound (25) for all $b > 0$. Imposing the additional constraint that $f_1, \ldots, f_n$ are multiples of $1/M$ yields Theorem 10.

For doubly-bounded-weight codes, we maintain the constraint $f_1 + \cdots + f_n = w$ and also require $f_1 + \cdots + f_{n_1} \leqslant w_1$. Again, the maximization can be carried out in either the continuous domain $[0, 1]$ or in the discrete domain $\{0, 1/M, 2/M, \ldots, 1\}$. This yields Theorem 29 in the discrete case and a weaker bound in the continuous case.

Relevant constraints for doubly-constant-weight codes are $f_1 + \cdots + f_{n_1} = w_1$ and $f_{n_1+1} + \cdots + f_{n_1+n_2} = w_2$. The resulting bounds are similar to (28) in the continuous case and to Theorem 29 in the discrete case. Both were proposed independently by Levenshtein [39] and by Johnson [37]. However, neither of them produces any improvement over the selection of bounds on doubly-constant-weight codes that is presented in Section VI.

IV. Bounds on $A(n, d, w)$

In this section, we summarize all important bounds on the cardinality of constant-weight codes that are known to us. Corollary 5 gives one such bound, but many more exist.

A. Elementary Bounds

The first theorem states without proof some elementary properties of $A(n, d, w)$.

Theorem 8:

$$A(n,d,w) = A(n,d+1,w), \qquad \text{if } d \text{ is odd} \tag{34}$$

$$A(n,d,w) = A(n,d,n-w) \tag{35}$$

$$A(n,2,w) = \binom{n}{w} \tag{36}$$

$$A(n,2w,w) = \left\lfloor \frac{n}{w} \right\rfloor \tag{37}$$

$$A(n,d,w) = 1, \qquad \text{if } d > 2w. \tag{38}$$

Example 5: $A(16, 10, 11) = A(16, 10, 5) = 3$ (to be continued in Example 16). □

The following theorem is due to Johnson [35].

Theorem 9:

$$A(n,d,w) \leqslant \left\lfloor \frac{n}{w} A(n-1,d,w-1) \right\rfloor, \qquad \text{if } w > 0$$

$$A(n,d,w) \leqslant \left\lfloor \frac{n}{n-w} A(n-1,d,w) \right\rfloor, \qquad \text{if } w < n.$$

The next theorem is equivalent to another of Johnson's bounds [35, eq. (6)], although it may look very different. Inspired by [39], we have formulated this theorem in a fashion that makes the relation to Proposition 7 apparent and highlights the symmetry between w and $n-w$. A proof was outlined in Section III-C.

Theorem 10: If $b > 0$, then

$$A(n, 2\delta, w) \leqslant \left\lfloor \frac{\delta}{b} \right\rfloor$$

where

$$b = \delta - \frac{w(n-w)}{n} + \frac{n}{M^2} \left\{ M\frac{w}{n} \right\} \left\{ M\frac{n-w}{n} \right\} \tag{39}$$

$$M = A(n, 2\delta, w) \tag{40}$$

$$\{x\} = x - \lfloor x \rfloor. \tag{41}$$

The foregoing upper bound on $A(n, d, w)$ is implicit since the quantity b depends on $A(n, d, w)$ through its dependence on M. Specifically, Theorem 10 implies that certain values of $A(n, d, w)$ are ruled out because they yield a contradiction. If an upper bound on $A(n, d, w)$ has this property, one can decrease the bound by 1 and try again.

Sometimes, when Theorem 10 holds with equality, it can be sharpened. This was done in two cases in [17]—see Example 6 for one of them. The next theorem details when, in general, such improvement is possible. This general result, to the best of our knowledge, is new.

Theorem 11: Suppose that $A(n, 2\delta, w) = \delta/b$, where b is given by (39). Then

$$A(n, 2\delta, w) \leqslant T(w_1, n_1, w_2, n_2, 2\delta)$$

where

$$w_1 = \frac{n_1}{n}\left(w - \frac{n_2}{M}\right) \tag{42}$$

$$n_1 = n - n\left\{M\frac{w}{n}\right\} \tag{43}$$

$$w_2 = \frac{n_2}{n}\left(w + \frac{n_1}{M}\right) \tag{44}$$

$$n_2 = n\left\{M\frac{w}{n}\right\} \tag{45}$$

and $M = A(n, 2\delta, w)$.

Proof: With n_1 and n_2 as defined in (43) and (45), we can rewrite (39) as

$$b = \delta - \frac{w(n-w)}{n} + \frac{n_1 n_2}{M^2 n}. \tag{46}$$

Let $\mathcal{C}$ be an $(n, 2\delta, w)$ constant-weight code, and assume that $\mathcal{C}$ contains $M = \delta/b$ codewords. This assumption imposes strong constraints on the structure of $\mathcal{C}$. First, according to Theorem 10, the bound in (32) must hold with equality, and we get

$$b = \delta - \sum_{i=1}^{n} f_i(1 - f_i)$$

which implies

$$\sum_{i=1}^{n} f_i(1 - f_i) = \frac{w(n-w)}{n} - \frac{n_1 n_2}{M^2 n} \tag{47}$$

in view of (46). Observe that $w(n-w)/n$ is the maximum value of the sum on the left-hand side of (47) subject to the constraint $f_1 + \cdots + f_n = w$. This value is attained when $f_i = w/n$ for all i. Subject to the additional constraint that $f_1, \ldots, f_n$ are multiples of $1/M$, we find that equality in (47) is possible if and only if

$$f_i = \frac{\left\lfloor M\frac{w}{n}\right\rfloor}{M} = \frac{w}{n} - \frac{n_2}{Mn}, \qquad \text{for } i \leqslant n_1 \tag{48}$$

$$f_i = \frac{\left\lfloor M\frac{w}{n}\right\rfloor + 1}{M} = \frac{w}{n} + \frac{n_1}{Mn}, \qquad \text{for } i > n_1 \tag{49}$$

up to permutations of the same sequence $f_1, \ldots, f_n$. Furthermore, a necessary condition for equality in (32) is that $d_{\text{av}} = 2\delta$, where d_{av} is as defined in (33). This means that *all* pairwise distances within the code are exactly 2δ, which in turn implies that every two codewords of $\mathcal{C}$ intersect in exactly $w - \delta$ positions.

Consider a codeword $\boldsymbol{c} = (c_1, \ldots, c_n) \in \mathcal{C}$. Let $w_1(\boldsymbol{c})$ and $w_2(\boldsymbol{c})$ denote the weights of the first n_1 and the last n_2 positions of $\boldsymbol{c}$, respectively. Let

$$\mathcal{W}(\boldsymbol{c}) = M \sum_{i \in \operatorname{supp}(\boldsymbol{c})} f_i$$

where $\operatorname{supp}(\boldsymbol{c})$ is the support of $\boldsymbol{c}$. Then

$$\mathcal{W}(\boldsymbol{c}) = w_1(\boldsymbol{c})\left\lfloor M\frac{w}{n}\right\rfloor + w_2(\boldsymbol{c})\left(\left\lfloor M\frac{w}{n}\right\rfloor + 1\right) \tag{50}$$

$$= w + (M-1)(w - \delta) \tag{51}$$

where (50) follows from (48) and (49), while (51) follows from the fact that every two codewords of $\mathcal{C}$ intersect in $w - \delta$ positions. Since $w_1(\boldsymbol{c}) + w_2(\boldsymbol{c}) = w$, (50) and (51) can be solved for the values of $w_1(\boldsymbol{c})$ and $w_2(\boldsymbol{c})$, which means that these values are independent of the choice of $\boldsymbol{c}$.

This proves that $\mathcal{C}$ is actually a doubly-constant-weight code. To find the values of $w_1(\boldsymbol{c}) = w_1$ and $w_2(\boldsymbol{c}) = w_2$, we first use the condition $M = \delta/b$ in conjunction with (46) to express δ as a function of M, w, n, n_1, and n_2. Substituting this expression into (51) leads to the solutions for w_1 and w_2 that are given by (42) and (44), respectively. ■

Example 6: From Corollary 5, we get $A(21, 10, 7) \leqslant 15$. Furthermore, $A(21, 10, 7) \neq 15$ by Theorem 21. Assume that $A(21, 10, 7) = 14$. Then Theorem 11 yields

$$A(21, 10, 7) \leqslant T(2, 7, 5, 14, 10).$$

But $T(2, 7, 5, 14, 10) \leqslant 13$ from Theorems 27 and 33, which is a contradiction. Hence $A(21, 10, 7) \leqslant 13$, which, in fact, holds with equality [42, p. 689]. □

We next describe another well-known upper bound on $A(n, 2\delta, w)$. In this context, let $t = w - \delta + 1$. A t-tuple is any subset of $\{1, \ldots, n\}$ of size t. Let $\mathcal{C}$ be an $(n, 2\delta, w)$ constant-weight code. We say that a given t-tuple is *covered* by a codeword $\boldsymbol{c} \in \mathcal{C}$ if it is a subset of the support of $\boldsymbol{c}$. It is easy to see that no t-tuple can be covered by two distinct codewords $\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathcal{C}$, since, otherwise, $d(\boldsymbol{c}_1, \boldsymbol{c}_2) < 2\delta$. The total number of t-tuples is $\binom{n}{t}$, and $\binom{w}{t}$ of these are covered by each codeword of $\mathcal{C}$. Thus we have proved the following.

Theorem 12: Let $t = w - \delta + 1$. Then

$$A(n, 2\delta, w) \leqslant \mathcal{X}(n, \delta, w) \stackrel{\text{def}}{=} \frac{\binom{n}{t}}{\binom{w}{t}}.$$

Theorem 12 also follows by recursive application of Theorem 9. The codewords of any code $\mathcal{C}$ that meets the bound of Theorem 12 with equality form a Steiner system $S(t, n, w)$. This means that every t-tuple is covered by exactly one codeword of $\mathcal{C}$. See [42, pp. 58–64, 528] and [53, pp. 1–4, 99–100] for more background on this topic.

If $\mathcal{X}(n, \delta, w)$ is an integer and it is known that a Steiner system $S(t, n, w)$ does not exist, the bound of Theorem 12 can be improved to $\mathcal{X}(n, \delta, w) - 1$. The next theorem makes it possible to further improve this bound to $\mathcal{X}(n, \delta, w) - 2$ under a certain condition. Although two special cases of this theorem were implicitly used in [8] (one such case is Example 7), the general result, to our knowledge, has not been previously published.

Theorem 13: If n divides $w\mathcal{X}(n, \delta, w)$, then

$$A(n, 2\delta, w) \neq \mathcal{X}(n, \delta, w) - 1.$$

Proof: Assume that $A(n, 2\delta, w) = \mathcal{X}(n, \delta, w) - 1$, and let $\mathcal{C}$ be a code that attains this bound. Note that this assumption implies, in particular, that $\mathcal{X}(n, \delta, w)$ is an integer. For all $i = 1, \ldots, n$, we have

$$|\mathcal{C}| f_i \leqslant \frac{\binom{n-1}{t-1}}{\binom{w-1}{t-1}} = \frac{w\mathcal{X}(n, \delta, w)}{n} \tag{52}$$

since, otherwise, there exists a t-tuple, involving position i, that is covered by two codewords. On the other hand,

$$|\mathcal{C}| \sum_{i=1}^{n} f_i = |\mathcal{C}| w = w\mathcal{X}(n, \delta, w) - w \tag{53}$$

by assumption. This implies that (52) must hold with equality for at least $n-w$ values of i. Without loss of generality, let these values be $i = 1, \dots, n-w$. This means that every t-tuple that involves any of the first $n-w$ positions is covered by a codeword of $\mathcal{C}$. The total number of such t-tuples is $\binom{n}{t} - \binom{w}{t}$. Since $|\mathcal{C}| = \mathcal{X}(n, \delta, w) - 1$ by assumption, this is precisely equal to the total number $(\mathcal{X}(n, \delta, w) - 1)\binom{w}{t}$ of t-tuples covered by the codewords of $\mathcal{C}$. This, in turn, implies that none of the $\binom{w}{t}$ t-tuples that involve only the last w positions is covered by a codeword of $\mathcal{C}$. A vector $\boldsymbol{x} = (0, \dots, 0, 1, \dots, 1)$ of weight w covers all these t-tuples and no others. Hence $\mathcal{C}' = \mathcal{C} \cup \{\boldsymbol{x}\}$ is an $(n, 2\delta, w)$ constant-weight code. This contradicts the assumption that $A(n, 2\delta, w) = \mathcal{X}(n, \delta, w) - 1$. ■

Example 7: Consider the case $(n, 2\delta, w) = (15, 4, 5)$. Then $\mathcal{X}(n, \delta, w) = 273$, which is not achievable by Theorem 21. Since 15 divides $5 \cdot 273$, the condition of Theorem 13 holds, and the theorem proves that $A(15, 4, 5)$ cannot equal 272 either. Hence $A(15, 4, 5) \leqslant 271$, which was stated without proof in [8] (though $A(15, 4, 5) \leqslant 272$ was proved there). □

B. The Freiman–Berger–Johnson Bound

The well-known Hamming bound [33] for unrestricted codes is obtained by centering a sphere around each codeword. Johnson [37] developed a family of bounds for constant-weight codes using a similar technique, and thereby generalized a bound by Berger [7], who in turn generalized a bound by Freiman [30].

Johnson [37] gives a range of versions of the same general bound, which leaves the user of these bounds some freedom to choose a suitable level of complexity. Since the original presentation in [37] does not contain an explicit description on how to evaluate these bounds, we now summarize the key equations necessary for complete implementation.

Theorem 14: For all $j = -\delta, -\delta+1, \dots, \delta$, we have

$$A(n, 2\delta, w) \leqslant \left\lfloor \frac{\binom{n}{w-j}}{\mathcal{K}(n,\delta,w,j) + \mathcal{L}(n,\delta,w,j)} \right\rfloor$$

where

$$\mathcal{L}(n, \delta, w, j) \stackrel{\text{def}}{=} \sum_{i=\max\{0,j\}}^{\lfloor(\delta+j-1)/2\rfloor} \binom{w}{i}\binom{n-w}{i-j}$$

while the value of $\mathcal{K}(n, \delta, w, j)$ depends on the parity of j as follows. If $j \equiv \delta \bmod 2$, then

$$\mathcal{K}(n,\delta,w,j) \stackrel{\text{def}}{=} \max\left\{\frac{A}{\alpha}, \frac{2\beta A - B}{\beta(1+\beta)}\right\}$$

where

$$\begin{aligned}
\beta &\stackrel{\text{def}}{=} \left\lfloor 1 + \frac{B}{A} \right\rfloor \\
B &\stackrel{\text{def}}{=} \binom{\delta}{\gamma}^2 T(\delta, w, \delta, n-w, 2\delta) \\
A &\stackrel{\text{def}}{=} \binom{w}{j+\gamma}\binom{n-w}{\gamma} \\
\alpha &\stackrel{\text{def}}{=} \left\lfloor \frac{n-w+j}{\delta-\gamma} \right\rfloor, \qquad \text{if } \gamma = 0 \text{ or } \gamma < \frac{(w-j)\delta}{n} \\
\alpha &\stackrel{\text{def}}{=} \left\lfloor \frac{w-j}{\gamma} \right\rfloor, \qquad \text{if } \gamma > 0 \text{ and } \gamma \geqslant \frac{(w-j)\delta}{n} \\
\gamma &\stackrel{\text{def}}{=} \frac{\delta - j}{2}.
\end{aligned}$$

If $j \not\equiv \delta \bmod 2$, then

$$\mathcal{K}(n,\delta,w,j) \stackrel{\text{def}}{=} \max\left\{0, \frac{A}{\alpha}\right\}$$

where

$$\begin{aligned}
A &\stackrel{\text{def}}{=} \binom{w}{j+\gamma+1}\binom{n-w}{\gamma+1} - \binom{\delta}{\gamma}\binom{\delta}{j+\gamma} T(\delta, w, \delta, n-w, 2\delta) \\
\alpha &\stackrel{\text{def}}{=} T(\gamma+1, w-j, j+\gamma+1, n-w+j, 2\delta) \\
\gamma &\stackrel{\text{def}}{=} \frac{\delta-j-1}{2}.
\end{aligned}$$

Theorem 14 specifies one version of the bounds in [37], namely, the same version that Johnson used in his experiments in that paper. Colbourn [20] successfully evaluated another, simpler, version. We have simplified the original notation of [37] for brevity and ease of reading.

C. Linear Programming

The *distance distribution* of a code $\mathcal{C} \subseteq \mathcal{H}(n)$ may be defined as

$$A_i \stackrel{\text{def}}{=} \frac{1}{|\mathcal{C}|} \sum_{\boldsymbol{c} \in \mathcal{C}} |\mathcal{S}_i(\boldsymbol{c})| \tag{54}$$

for $i = 0, \dots, n$, where $\mathcal{S}_i(\boldsymbol{c})$ denotes the shell of Hamming radius i centered at $\boldsymbol{c}$, namely,

$$\mathcal{S}_i(\boldsymbol{c}) \stackrel{\text{def}}{=} \{\boldsymbol{x} \in \mathcal{C}: d(\boldsymbol{c}, \boldsymbol{x}) = i\}.$$

The shell $\mathcal{S}_i(\boldsymbol{c})$ is equivalent under translation by $\boldsymbol{c}$ to a constant-weight code. If $\mathcal{C}$ is a constant-weight code, then $\mathcal{S}_i(\boldsymbol{c})$ is equivalent under translation and permutation to a doubly-constant-weight code.

The linear programming bound for constant-weight codes is based on the properties of the distance distribution of a code $\mathcal{C} \in \Phi(\mathcal{H}(n, w), 2\delta)$ for given constants n, w, and δ. Throughout this subsection, it is assumed that $w \leqslant n/2$. The component A_i of the distance distribution is, in this case, trivially zero for $i < 2\delta$, $i > 2w$, and whenever i is odd. Thus we focus on $A_{2\delta}, A_{2\delta+2}, \dots, A_{2w}$. The general idea is to find linear inequalities involving these components, for use in the linear programming problem of Theorem 20.

Since $\mathcal{S}_i(\boldsymbol{c})$ is a doubly-constant-weight code, its size can be upper-bounded as

$$|\mathcal{S}_i(\boldsymbol{c})| \leqslant T(i/2, w, i/2, n-w, 2\delta). \tag{55}$$

Combining this result with (54) yields the following well-known constraint [8].

Proposition 15: For all $i = \delta, \dots, w$

$$0 \leqslant A_{2i} \leqslant T(i, w, i, n-w, 2\delta).$$

The following profound inequality of Delsarte [24, Sec. 4.2] has led to the success of linear programming bounds for constant-weight codes.

Proposition 16: For all $k = 1, \ldots, w$

$$\sum_{i=\delta}^{w} q(k,i,n,w) A_{2i} \geqslant -1$$

where

$$q(k,i,n,w) \stackrel{\text{def}}{=} \frac{\sum_{j=0}^{i} (-1)^j \binom{k}{j} \binom{w-k}{i-j} \binom{n-w-k}{i-j}}{\binom{w}{i} \binom{n-w}{i}}. \tag{56}$$

It is known that the distance distribution of constant-weight codes is subject to more constraints than can be obtained from Propositions 15 and 16. However, determining these additional constraints has, in most cases, involved a separate nontrivial argument for each distinct set of parameters n, d, and w (as in [8, Theorem 22]). The following proposition is, in some sense, a generalization of this type of constraints. This proposition provides a universal method to find constraints for pairs of distance distribution components, given bounds on doubly-bounded-weight codes and doubly-constant-weight codes.

Proposition 17: Let $i, j \in \{\delta, \delta+1, \ldots, w\}$, with $i \neq j$. If $i + j > n - \delta$, then

$$P_j A_{2i} + P_i A_{2j} \leqslant P_i P_j \tag{57}$$

where P_i and P_j are any nonnegative integers such that

$$P_i \geqslant T(i, w, i, n-w, 2\delta) \tag{58}$$

$$P_j \geqslant T(j, w, j, n-w, 2\delta). \tag{59}$$

If $i + j \leqslant n - \delta$, define P_{ij} and P_{ji} as any nonnegative integers such that

$$P_{ij} \geqslant \min\{P_i, T'(\Delta, j, i-\Delta, n-w-j, 2i-2\Delta)\} \tag{60}$$

$$P_{ji} \geqslant \min\{P_j, T'(\Delta, i, j-\Delta, n-w-i, 2j-2\Delta)\} \tag{61}$$

where $\Delta \stackrel{\text{def}}{=} w - \delta$. Then

$$P_{ji} A_{2i} + (P_i - P_{ij}) A_{2j} \leqslant P_i P_{ji}, \quad \text{if } \frac{P_{ij}}{P_i} + \frac{P_{ji}}{P_j} > 1 \tag{62}$$

$$(P_j - P_{ji}) A_{2i} + P_{ij} A_{2j} \leqslant P_j P_{ij}, \quad \text{if } \frac{P_{ij}}{P_i} + \frac{P_{ji}}{P_j} > 1 \tag{63}$$

$$P_j A_{2i} + P_i A_{2j} \leqslant P_i P_j, \quad \text{if } \frac{P_{ij}}{P_i} + \frac{P_{ji}}{P_j} \leqslant 1. \tag{64}$$

Proof: The proof relies on the following lemma that relates the sizes of two shells $\mathcal{S}_{2i}(\boldsymbol{c})$ and $\mathcal{S}_{2j}(\boldsymbol{c})$.

Lemma 18: Let $i, j \in \{\delta, \ldots, w\}$, with $i \neq j$; and let $\boldsymbol{c} \in \mathcal{C}$. If $|\mathcal{S}_{2i}(\boldsymbol{c})| \geqslant 1$, then

$$|\mathcal{S}_{2j}(\boldsymbol{c})| \leqslant T'(\Delta, i, j-\Delta, n-w-i, 2j-2\Delta)$$

for $j \leqslant n - \delta - i$, and $\mathcal{S}_{2j}(\boldsymbol{c}) = \varnothing$ elsewhere.

Proof: Let $\boldsymbol{x} \in \mathcal{S}_{2i}(\boldsymbol{c})$. Without loss of generality, reorder the positions so that $\boldsymbol{c}$ and $\boldsymbol{x}$ have the forms

$$\boldsymbol{c} = (\overbrace{1, \ldots, 1, 1, \ldots, 1}^{w}, \overbrace{0, \ldots, 0, 0, \ldots, 0}^{n-w}) \tag{65}$$

$$\boldsymbol{x} = (\underbrace{1, \ldots, 1}_{w-i}, \underbrace{0, \ldots, 0}_{i}, \underbrace{1, \ldots, 1}_{i}, \underbrace{0, \ldots, 0}_{n-w-i}). \tag{66}$$

If $\mathcal{S}_{2j}(\boldsymbol{c}) = \varnothing$, there is nothing to prove. Otherwise, consider any codeword $\boldsymbol{y} \in \mathcal{S}_{2j}(\boldsymbol{c})$. As in (66), it must have j zeros among the first w positions and j ones among the last $n - w$ positions. Let $d_1(\cdot)$ and $d_2(\cdot)$ denote the Hamming distance between the first w positions and the last $n - w$ positions of two codewords, respectively.Then $d_1(\boldsymbol{y}, \boldsymbol{x}) \leqslant (w-i) + (w-j)$. Since $d(\boldsymbol{y}, \boldsymbol{x}) \geqslant 2\delta$, we have

$$d_2(\boldsymbol{y}, \boldsymbol{x}) = d(\boldsymbol{y}, \boldsymbol{x}) - d_1(\boldsymbol{y}, \boldsymbol{x}) \geqslant i + j - 2\Delta.$$

This implies that $\boldsymbol{y}$ has at least $j - \Delta$ ones among the last $n - w - i$ positions and at most Δ ones in the preceding block of i positions. (If $i + j > n - \delta$, this is impossible, and hence $\mathcal{S}_{2j}(\boldsymbol{c})$ must be empty.) It follows that the punctured code obtained by extracting the last $n - w$ positions from $\mathcal{S}_{2j}(\boldsymbol{c})$ is a doubly-bounded-weight code. To bound its distance, consider any pair of codewords $\boldsymbol{y}, \boldsymbol{z}$ in $\mathcal{S}_{2j}(\boldsymbol{c})$. They satisfy $d_1(\boldsymbol{y}, \boldsymbol{z}) \leqslant 2w - 2j$, and hence $d_2(\boldsymbol{y}, \boldsymbol{z}) \geqslant 2j - 2\Delta$. ∎

Remark: Although Lemma 18 is valid for any distinct $i, j \in \{\delta, \ldots, w\}$, parameters near the lower end of this interval yield useless bounds. In particular, it follows from the results of Sections V-A and VI-A that if $i \leqslant \Delta$, then

$$T'(\delta, i, j-\Delta, n-w-i, 2j-2\Delta) \geqslant T(j, w, j, n-w, 2\delta).$$

Hence Lemma 18 gives a weaker bound on $|\mathcal{S}_{2j}(\boldsymbol{c})|$ than (55) whenever $i \leqslant w - \delta$. Thus the application of Lemma 18 can be confined to $i, j \geqslant \max\{\delta, \Delta + 1\}$.

We are now ready to complete the proof of Proposition 17. It follows from (55) and Lemma 18 that

$$|\mathcal{S}_{2i}(\boldsymbol{c})| \leqslant P_i \tag{67}$$

$$|\mathcal{S}_{2i}(\boldsymbol{c})| \leqslant P_{ij}, \qquad \text{if } |\mathcal{S}_{2j}(\boldsymbol{c})| > 0 \tag{68}$$

$$|\mathcal{S}_{2j}(\boldsymbol{c})| \leqslant P_j \tag{69}$$

$$|\mathcal{S}_{2j}(\boldsymbol{c})| \leqslant P_{ji}, \qquad \text{if } |\mathcal{S}_{2i}(\boldsymbol{c})| > 0 \tag{70}$$

with P_i, P_{ij}, P_j, and P_{ji} as in (58)–(61). Define the sets

$$\mathcal{C}_i \stackrel{\text{def}}{=} \{\boldsymbol{c} \in \mathcal{C} \colon \mathcal{S}_{2i}(\boldsymbol{c}) > 0\}$$

$$\mathcal{C}_j \stackrel{\text{def}}{=} \{\boldsymbol{c} \in \mathcal{C} \colon \mathcal{S}_{2j}(\boldsymbol{c}) > 0\}.$$

Then (54), in conjunction with (67) and (68), yields

$$|\mathcal{C}| A_{2i} = \sum_{\boldsymbol{c} \in \mathcal{C}_i \cap \overline{\mathcal{C}_j}} |\mathcal{S}_{2i}(\boldsymbol{c})| + \sum_{\boldsymbol{c} \in \mathcal{C}_i \cap \mathcal{C}_j} |\mathcal{S}_{2i}(\boldsymbol{c})|$$

$$\leqslant |\mathcal{C}_i \cap \overline{\mathcal{C}_j}| P_i + |\mathcal{C}_i \cap \mathcal{C}_j| P_{ij} \tag{71}$$

where $\overline{\mathcal{A}}$ denotes the complement of a set $\mathcal{A}$. Similarly, we have

$$|\mathcal{C}| A_{2j} \leqslant |\overline{\mathcal{C}_i} \cap \mathcal{C}_j| P_j + |\mathcal{C}_i \cap \mathcal{C}_j| P_{ji}. \tag{72}$$

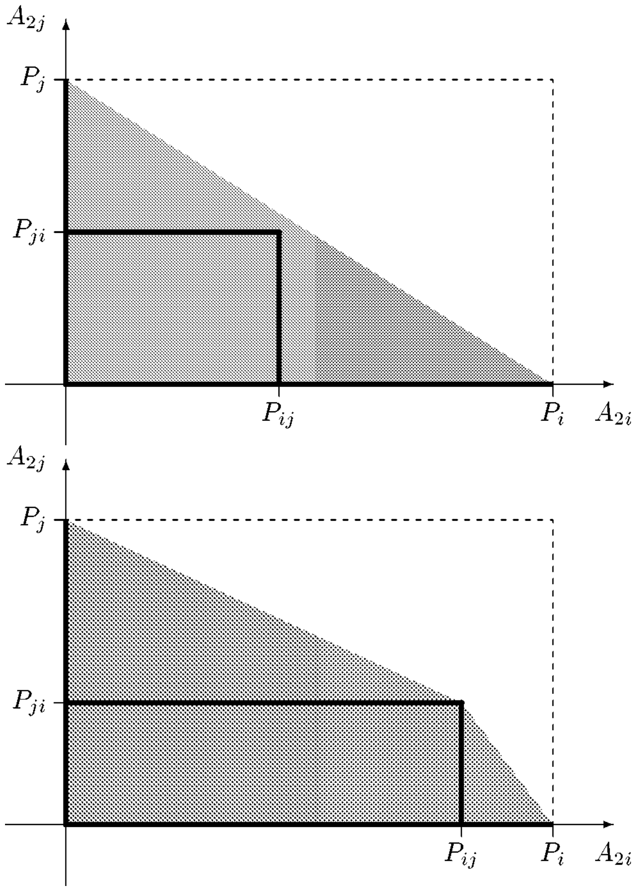

Fig. 3. The inequalities (67)–(70) define the region enclosed by the thick lines. Its convex hull (shaded) is the domain of (A_{2i}, A_{2j}). Dashed lines indicate the well-known bound of Proposition 15.

We multiply both sides of (71) by $P_j - P_{ji}$ and both sides of (72) by P_{ij}. Adding the results then yields

$$\begin{aligned}(P_j - P_{ji})|\mathcal{C}|A_{2i} + P_{ij}|\mathcal{C}|A_{2j} \\ &\leqslant |\mathcal{C}_i \cup \mathcal{C}_j|P_jP_{ij} - |\mathcal{C}_i \cap \overline{\mathcal{C}_j}|(P_iP_{ji} + P_jP_{ij} - P_iP_j) \\ &\leqslant |\mathcal{C}|P_jP_{ij}, \qquad \text{if } P_iP_{ji} + P_jP_{ij} - P_iP_j \geqslant 0\end{aligned}$$

where we have used some elementary set relations to establish the first inequality. This proves (63). The bound (62) follows by symmetry. To prove (64), we take a different linear combination of (71) and (72), namely,

$$\begin{aligned}P_j|\mathcal{C}|A_{2i} + P_i|\mathcal{C}|A_{2j} \\ &\leqslant |\mathcal{C}_i \cup \mathcal{C}_j|P_iP_j - |\mathcal{C}_i \cap \mathcal{C}_j|(P_iP_j - P_iP_{ji} - P_jP_{ij}).\end{aligned}$$

Finally, the bound (57) for $i+j > n-\delta$ follows from the above by observing that $\mathcal{C}_i \cap \mathcal{C}_j$ is empty in this case. ■

From a geometrical viewpoint, the inequalities (67)–(70) can be regarded as lines bounding a region in the plane. Two examples are shown in Fig. 3. The definition of the distance distribution in (54) implies that a point (A_{2i}, A_{2j}) is formed by averaging the points $(|\mathcal{S}_{2i}(\boldsymbol{c})|, |\mathcal{S}_{2j}(\boldsymbol{c})|)$ for all $\boldsymbol{c} \in \mathcal{C}$. Hence the domain of (A_{2i}, A_{2j}) is the convex hull of the domain of $(|\mathcal{S}_{2i}(\boldsymbol{c})|, |\mathcal{S}_{2j}(\boldsymbol{c})|)$. This convex hull is a polygon with either three or four sides, depending on the values of P_i, P_j, P_{ij}, and P_{ji}. This is illustrated in Fig. 3 (top) and (bottom), respectively. In the former case, the polygon is bounded by (64) and in the latter case by (62) and (63). Note that if $P_{ij} = P_i$ and $P_{ji} = P_j$ in (62) and (63), then the polygon becomes a rectangle and Proposition 17 reduces to Proposition 15. In all other cases, Proposition 17 gives a stronger constraint on the distance distribution $A_{2\delta}, A_{2\delta+2}, \ldots, A_{2w}$ than Proposition 15.

Remark: It would suffice to evaluate Proposition 17 for i and j such that $\max\{\delta, \Delta+1\} \leqslant j < i \leqslant w$. The lower bound comes from the earlier remark regarding Lemma 18, while $i > j$ can be assumed without loss of generality.

Example 8: Suppose that $(n, d, w) = (27, 10, 11)$, and consider $(i, j) = (11, 10)$. We have

$$\begin{aligned}P_i &= 3 = T(11, 11, 11, 16, 10) \\ P_j &= 13 \geqslant T(10, 11, 10, 16, 10) \\ P_{ij} &= 2 = T'(6, 10, 5, 6, 10) \\ P_{ji} &= 7 \geqslant T'(6, 11, 4, 5, 8)\end{aligned}$$

from Examples 16, 17, 11, and 13, respectively. Then Proposition 17 yields $A_{20} + 7A_{22} \leqslant 21$ and $A_{20} + 3A_{22} \leqslant 13$. This example will be concluded in Example 10. □

The following proposition gives another useful constraint on the distance distribution of constant-weight codes derived from bounds for doubly-bounded-weight codes.

Proposition 19: For all $j = \delta, \delta+1, \ldots, w-1$, we have

$$\sum_{i=j}^{w} A_{2i} \leqslant T'(w-j, w, j, n-w, 2\delta). \tag{73}$$

Proof: For any code $\mathcal{C} \in \Phi(\mathcal{H}(n, w), 2\delta)$ and any codeword $\boldsymbol{c} \in \mathcal{C}$, the set $\bigcup_{i=j}^{w} \mathcal{S}_{2i}(\boldsymbol{c})$ is a doubly-bounded-weight code with parameters as in (73). ■

Having established the constraints on the distance distribution, we now state the linear programming bound itself.

Theorem 20: If $w \leqslant n/2$, then

$$A(n, 2\delta, w) \leqslant \left\lfloor \max \sum_{i=\delta}^{w} A_{2i} \right\rfloor + 1$$

where the maximum is taken over all $(A_{2\delta}, A_{2\delta+2}, \ldots, A_{2w})$ that satisfy the constraints in Propositions 15–17 and 19.

Example 9: For $(n, d, w) = (20, 8, 9)$, the linear programming bound, using the constraints developed in Propositions 17 and 19, yields $A(20, 8, 9) \leqslant 195$. This improves upon the best previously known upper bound of 215. □

Example 10: Using the constraints on A_{20} and A_{22} derived in Example 8, linear programming yields the upper bound $A(27, 10, 11) \leqslant 900$. □

D. Specific Bounds

In this subsection, bounds that hold only for specific values of n, d, and w are collected and discussed. The following theorem lists all the relevant specific bounds that we are aware of. This theorem does not include all specific bounds that have ever been proposed; some of them have later been reproduced or superseded by general bounds.

Theorem 21:

$$A(15,4,5) \leqslant 272 \quad (74)$$
$$A(13,6,5) = 18 \quad (75)$$
$$A(14,6,7) = 42 \quad (76)$$
$$A(17,6,4) = 20 \quad (77)$$
$$A(18,6,6) \leqslant 203 \quad (78)$$
$$A(19,6,4) = 25 \quad (79)$$
$$A(17,8,7) = 24 \quad (80)$$
$$A(21,8,10) \leqslant 399 \quad (81)$$
$$A(22,8,5) = 21 \quad (82)$$
$$A(23,8,10) \leqslant 1109 \quad (83)$$
$$A(26,8,5) = 30 \quad (84)$$
$$A(20,10,8) = 17 \quad (85)$$
$$A(21,10,7) \leqslant 14 \quad (86)$$
$$A(22,10,7) = 16 \quad (87)$$
$$A(22,10,10) \leqslant 73 \quad (88)$$
$$A(22,10,11) \leqslant 81 \quad (89)$$
$$A(23,10,7) = 20 \quad (90)$$
$$A(26,12,11) \leqslant 69 \quad (91)$$
$$A(26,12,12) \leqslant 83 \quad (92)$$
$$A(26,12,13) \leqslant 92 \quad (93)$$
$$A(27,12,10) \leqslant 65 \quad (94)$$
$$A(27,12,11) \leqslant 100 \quad (95)$$
$$A(28,12,8) \leqslant 20. \quad (96)$$

We have not verified all the values in Theorem 21. In general, it is very difficult to check specific upper bounds found by others. (As pointed out in [17], an extreme case of this is the celebrated result of Lam, Thiel, and Swiercz [38] that there is no projective plane of order 10, which is equivalent to $A(111, 20, 11) \leqslant 110$. The proof of [38] is based on years of research and thousands of hours of computer time.) Thus Theorem 21 relies on the published literature. We now provide references for each bound listed in Theorem 21.

The bounds (77) and (79) were obtained by Brouwer [16] and Stinson [51], respectively. The method used was assuming the existence of a code with a higher value of $A(n, d, w)$, identifying properties of this hypothetical code, and arriving at a contradiction. The bound (75) is given as a problem in [42, p. 531], where it is suggested that it can be proved using a similar technique.

The bounds (74) and (78) follow from the nonexistence of certain Steiner systems, while (86) and (96) follow from the nonexistence of certain 2-designs [21], [25], [32] (see [17] and the discussion following Theorem 12). These four bounds can each be decreased by one using Theorems 11 or 13.

The value in (84) was derived in [13] from the nonexistence of a certain instance of what is known as a partial linear space [18, pp. 68–70, 435, 650].

The bounds (88) and (89) were obtained by van Pul [45, p. 38] using linear programming with constraints specifically derived for these parameters. In a similar manner, Honkala [34, Sec. 5] obtained (91)–(95).

A full search algorithm by Brouwer, Shearer, Sloane, and Smith [17] has contributed several exact values of $A(n, d, w)$ compiled in [17, Table III]. The cases in Theorem 21 obtained from this source are (76), (80), (85), (87), and (90). See also Appendix A regarding the exact value of (90).

The bounds (81) and (83) have been reported as results of linear programming [42, p. 688] and the Freiman–Berger–Johnson bound [37], respectively. Both [42] and [37] apparently used undisclosed constraints to obtain these bounds.

Finally, the bound (82) is from [17, Table III], where the only justification is: "By the Bose–Connor theorem a square divisible design $GD(5, 1, 2; 11\times 2)$ does not exist." We believe it would be useful to provide a more elaborate argument, as follows. Let $(n, 2\delta, w) = (22, 8, 5)$ and proceed in a manner similar to the proof of Theorem 13. From (52), we have $|\mathcal{C}|f_i \leqslant 5$ for all i, which implies that

$$|\mathcal{C}| = \sum_{i=1}^{n} \frac{|\mathcal{C}|f_i}{w} \leqslant 22.$$

For a code that attains this bound, we must have $|\mathcal{C}|f_i = 5$ for all i. The t-tuples in this case are simply pairs, and out of all $\binom{22}{2} = 231$ pairs of positions, 220 are covered by the 22 codewords, in such a way that each position is contained in exactly 20 covered pairs. It follows that the remaining 11 pairs, which are not covered by any codeword, are disjoint. This structure is known, in the terminology of design theory, as a group divisible incomplete block design with parameters

$$(v, r, b, k, m, n, \lambda_1, \lambda_2) = (22, 5, 22, 5, 11, 2, 0, 1)$$

but no such design exists [14].

E. Redundant Bounds

Many bounds for constant-weight codes have been proposed, but not all of them remain competitive today. Our intent in this work is to list *all* the upper bounds for constant-weight codes known to us. Thus for completeness, we briefly mention in this section those bounds that were evaluated in the present study but did not contribute to our tables in Section VII.

Two standard bounds that we have so far omitted are [42, p. 525, Theorems 1(d), 2]. As already mentioned, both are contained in Corollary 5, and are often improved upon by this corollary. The upper bound version of [8, Theorem 20] also does not need to be separately considered. It can be shown that this theorem is weaker than Theorem 10.

Levenshtein's bound [39, eq. (4)] relates constant-weight codes to doubly-constant-weight codes in precisely the same way as the Bassalygo–Elias inequality (1) relates unrestricted codes to constant-weight codes. It yields, in conjunction with the linear programming bound of [43], the best known upper bound on $A(n, d, w)$ asymptotically, as n, d, and w tend to infinity [4], [5], [48]. Nevertheless, neither [39, eq. (4)] nor its strengthened version [39, eq. (5)] improve on any of the values in our tables. Neither does [41, Theorem 6.25], which has the same asymptotical performance. Apparently, $n \leqslant 28$ is not large enough to do these bounds justice.

Known results on Steiner systems yield exact values of $A(n, d, w)$ in a number of cases [35, p. 207], [8, p. 90], [17, pp. 1339, 1341]. All such values were also obtained by some other means in our investigation.

The linear programming bound suggested by van Pul [45, Sec. 3.3] was implemented, and we also combined his constraint [45, p. 20] with the constraints of Section IV-C. No improvements were obtained from this general approach, but including constraints specific for each instance of (n, d, w) has led to interesting results; see Section IV-D.

V. Bounds on $T'(w_1, n_1, w_2, n_2, d)$

All the bounds for doubly-bounded-weight codes derived here are new. Our motivation for introducing and studying these codes is that they have strong connections to constant-weight codes. Several methods for bounding the size of constant-weight codes based on doubly-bounded-weight codes, either directly or indirectly, via doubly-constant-weight codes, are presented in Sections IV and VI. These relations are also summarized in Fig. 1.

A. Elementary Bounds

As defined in Section II, a doubly-bounded-weight code is any subset of $\mathcal{H}'(w_1, n_1, w_2, n_2)$. Thus doubly-bounded-weight codes are a subclass of constant-weight codes obtained by imposing an upper bound on the weight of the head or a lower bound on the weight of the tail. Let

$$p \stackrel{\text{def}}{=} \min\{w_1, n_2 - w_2\}. \tag{97}$$

It follows immediately from the definition (3) that for any vector in $\mathcal{H}'(w_1, n_1, w_2, n_2)$, the weight of the head ranges from $w_1 - p$ to w_1, and the weight of the tail ranges, correspondingly, from w_2 to $w_2 + p$.

Since each of the relations in the following theorem is straightforward, we omit the proofs.

Theorem 22:

$$\begin{aligned}
T'(0, n_1, w_2, n_2, d) &= A(n_2, d, w_2)\\
T'(n_1, n_1, w_2, n_2, d) &= A(n_1 + n_2, d, n_1 + w_2)\\
T'(w_1, n_1, 0, n_2, d) &= A(n_1 + n_2, d, w_1)\\
T'(w_1, n_1, n_2, n_2, d) &= A(n_1, d, w_1)\\
T'(w_1, n_1, w_2, n_2, d) &= T'(n_2 - w_2, n_2, n_1 - w_1, n_1, d)\\
T'(w_1, n_1, w_2, n_2, d) &= T'(w_1, n_1, w_2, n_2, d+1),\\
&\qquad\qquad \text{if } d \text{ is odd}\\
T'(w_1, n_1, w_2, n_2, 2) &= \sum_{i=0}^{p} \binom{n_1}{w_1 - i}\binom{n_2}{w_2 + i}.
\end{aligned}$$

In the following cases, simple expressions exist for the exact value of $T'(w_1, n_1, w_2, n_2, d)$.

Theorem 23:

$$T'(w_1, n_1, w_2, n_2, 2(w_1 + w_2)) = \left\lfloor \frac{n_2}{w_2} \right\rfloor, \quad \text{if } \frac{w_1}{n_1} \leqslant \frac{w_2}{n_2} \tag{98}$$

$$T'(w_1, n_1, w_2, n_2, 2(w_1 + w_2)) = \left\lfloor \frac{n_1 + n_2}{w_1 + w_2} \right\rfloor, \quad \text{if } \frac{w_1}{n_1} \geqslant \frac{w_2}{n_2} \tag{99}$$

$$T'(w_1, n_1, w_2, n_2, 2(n_2 + w_1 - w_2)) = \left\lfloor \frac{n_2}{n_2 - w_2} \right\rfloor, \quad \text{if } \frac{w_1}{n_1} + \frac{w_2}{n_2} \leqslant 1 \tag{100}$$

$$T'(w_1, n_1, w_2, n_2, 2(n_2 + w_1 - w_2)) = \left\lfloor \frac{n_1}{w_1} \right\rfloor, \quad \text{if } \frac{w_1}{n_1} + \frac{w_2}{n_2} \geqslant 1 \tag{101}$$

$$T'(w_1, n_1, w_2, n_2, 2(n_1 + n_2 - w_1 - w_2)) = \left\lfloor \frac{n_1}{n_1 - w_1} \right\rfloor, \quad \text{if } \frac{w_1}{n_1} \leqslant \frac{w_2}{n_2} \tag{102}$$

$$T'(w_1, n_1, w_2, n_2, 2(n_1 + n_2 - w_1 - w_2)) = \left\lfloor \frac{n_1 + n_2}{n_1 + n_2 - w_1 - w_2} \right\rfloor, \quad \text{if } \frac{w_1}{n_1} \geqslant \frac{w_2}{n_2} \tag{103}$$

$$T'(w_1, n_1, w_2, n_2, d) = 1, \quad \text{if } d > 2\min\{w_1 + w_2, n_2 + w_1 - w_2, n_1 + n_2 - w_1 - w_2\}. \tag{104}$$

Proof: The distance between two codewords of a code in $\mathcal{H}'(w_1, n_1, w_2, n_2)$ equals $2w_1 + 2w_2$ if and only if their ones are in disjoint positions. The total number of codewords with disjoint ones is upper-bounded by $\lfloor (n_1 + n_2)/(w_1 + w_2) \rfloor$. Similarly, the total number of codewords with disjoint ones in the tails is upper-bounded by $\lfloor n_2/w_2 \rfloor$. Thereby the upper bound versions of (98) and (99) are proved. To prove that these bounds are attainable with equality, we consider two constructions. First, let $\mathcal{C}_1 \in \Phi(\mathcal{H}(n_1, w_1), 2w_1)$ and $\mathcal{C}_2 \in \Phi(\mathcal{H}(n_2, w_2), 2w_2)$ with $|\mathcal{C}_1| = |\mathcal{C}_2| = \lfloor n_2/w_2 \rfloor$. Such codes exist, according to (37), if $\lfloor n_2/w_2 \rfloor \leqslant \lfloor n_1/w_1 \rfloor$. The code formed by joining each codeword in $\mathcal{C}_1$ with a unique codeword in $\mathcal{C}_2$ belongs to $\Phi(\mathcal{H}'(w_1, n_1, w_2, n_2), 2w_1 + 2w_2)$, which proves (98). Now, let $\mathcal{C}$ be a code in $\Phi(\mathcal{H}(n_1 + n_2, w_1 + w_2), 2w_1 + 2w_2)$ with

$$|\mathcal{C}| = \lfloor (n_1 + n_2)/(w_1 + w_2) \rfloor.$$

Then reordering the positions so that all codewords have at most w_1 ones in their heads (which can be done if $w_1|\mathcal{C}| \leqslant n_1$) completes the proof of (99).

The proofs of the remaining cases, except (104), are similar. The distance between codewords whose ones in the heads and zeros in the tails are in disjoint positions is $2(n_2 + w_1 - w_2)$, and the distance between codewords with disjoint zeros in all positions is $2(n_1 + n_2 - w_1 - w_2)$. The details of these proofs are omitted. Finally, (104) follows from the foregoing two observations, along with the fact that the distance between two codewords cannot be greater than $2w_1 + 2w_2$. ■

Example 11: From (102), we have $T'(6, 10, 5, 6, 10) = 2$ (this example is continued in Example 8). □

The simple nature of the next bound may suggest that it is not very strong. It is, however, useful in certain cases, as demonstrated later in Example 12.

Theorem 24:

$$T'(w_1, n_1, w_2, n_2, d) \leqslant T'(w_1 + 1, n_1, w_2 - 1, n_2, d), \quad \text{if } w_1 < n_1 \text{ and } w_2 > 0 \tag{105}$$

$$T'(w_1, n_1, w_2, n_2, d) \leqslant T'(w_1, n_1 + 1, w_2, n_2, d) \tag{106}$$

$$T'(w_1, n_1, w_2, n_2, d) \leqslant T'(w_1{+}1, n_1{+}1, w_2, n_2, d) \tag{107}$$

$$T'(w_1, n_1, w_2, n_2, d) \leqslant T'(w_1, n_1, w_2, n_2 + 1, d) \tag{108}$$

$$T'(w_1, n_1, w_2, n_2, d) \leqslant T'(w_1, n_1, w_2{+}1, n_2{+}1, d). \tag{109}$$

Proof: The bound (105) is a consequence of the fact that $\mathcal{H}'(w_1, n_1, w_2, n_2) \subseteq \mathcal{H}'(w_1{+}1, n_1, w_2{-}1, n_2)$. Appending a zero or a one to all codewords of a doubly-bounded-weight code yields (106)–(109). ■

Theorem 25: Let p be as defined in (97). Then

$$T'(w_1, n_1, w_2, n_2, d) \leqslant T(w_1, n_1 + p, w_2 + p, n_2 + p, d).$$

Proof: Extending the head of a doubly-bounded-weight code with p bits, suitably chosen for each codeword, assures that the weight of the head is a constant w_1. An additional p extra bits make the weight of the tail $w_2 + p$. ■

Theorem 26: For all $i = 0, \ldots, p - 1$

$$\begin{aligned} T'(w_1, n_1, w_2, n_2, d) \leqslant{}& T'(w_1 - i - 1, n_1, w_2 + i + 1, n_2, d) \\ &+ T(w_1, n_1 + i, w_2 + i, n_2 + i, d). \end{aligned}$$

Proof: We partition a code in $\Phi(\mathcal{H}'(w_1, n_1, w_2, n_2), d)$ into two subcodes. Let the codewords with weight at most $w_1 - i - 1$ in the heads form one subcode and the remaining codewords form another. The former subcode belongs to $\mathcal{H}'(w_1 - i - 1, n_1, w_2 + i + 1, n_2)$. In the latter subcode, the weight in the heads ranges from $w_1 - i$ to w_1, and in the tails from w_2 to $w_2 + i$. Extending the latter code with $2i$ bits as in the proof of Theorem 25 yields a code in $\mathcal{H}(w_1, n_1{+}i, w_2{+}i, n_2{+}i)$. ■

Theorem 27:

$$T'(w_1, n_1, w_2, n_2, 2\delta) \leqslant A(n_1 + n_2, 2\delta, w_1 + w_2) \tag{110}$$

$$\begin{aligned} &T'(w_1, n_1, w_2, n_2, 2\delta) \\ &\quad \leqslant A(n_1 + n_2, 2\min\{\delta, n_1 - w_1, w_2\}, w_1 + w_2) - 1. \end{aligned} \tag{111}$$

The latter bound holds with equality if $\delta = n_1 - w_1 = w_2$.

Proof: The bound (110) is obvious from (3). To prove (111), let

$$\boldsymbol{c}' = (\underbrace{1, \ldots, 1}_{w_1 + w_2}, \underbrace{0, \ldots, 0}_{n_1 + n_2 - w_1 - w_2})$$

and consider a code $\mathcal{C} \in \Phi(\mathcal{H}'(w_1, n_1, w_2, n_2), 2\delta)$. Now $\mathcal{C}' = \mathcal{C} \cup \{\boldsymbol{c}'\}$ is a constant-weight code with $|\mathcal{C}'| = |\mathcal{C}| + 1$. Its minimum distance is

$$d(\mathcal{C}') = \min\left\{ d(\mathcal{C}), \min_{\boldsymbol{c} \in \mathcal{C}} d(\boldsymbol{c}, \boldsymbol{c}') \right\}. \tag{112}$$

To derive a lower bound on $d(\boldsymbol{c}, \boldsymbol{c}')$, we consider two cases. If $w_1 + w_2 \leqslant n_1$, then the tail of $\boldsymbol{c}'$ is all-zero, which implies that heads and tails each contribute at least w_2 to the distance. Analogously, if $w_1 + w_2 \geqslant n_1$, then the head of $\boldsymbol{c}'$ contains only ones, and heads and tails contribute at least $n_1 - w_1$ each to the distance. Hence

$$\begin{aligned} d(\boldsymbol{c}, \boldsymbol{c}') &\geqslant 2w_2, && \text{if } w_1 + w_2 \leqslant n_1 \\ d(\boldsymbol{c}, \boldsymbol{c}') &\geqslant 2n_1 - 2w_1, && \text{if } w_1 + w_2 \geqslant n_1 \end{aligned}$$

or, equivalently,

$$d(\boldsymbol{c}, \boldsymbol{c}') \geqslant 2\min\{n_1 - w_1, w_2\}$$

for all $\boldsymbol{c} \in \mathcal{C}$. Thus $d(\mathcal{C}') \geqslant 2\min\{\delta, n_1 - w_1, w_2\}$.

To prove the equality part of (111), consider any constant-weight code that attains $A(n, 2\delta, w)$. If we reorder the bits so that $(1, \ldots, 1, 0, \ldots, 0)$ is a codeword and then remove this codeword, then all of the remaining $A(n, 2\delta, w) - 1$ codewords have at least δ zeros in the first w positions. The doubly-bounded-weight code formed by these codewords demonstrates that

$$A(n, 2\delta, w) - 1 \leqslant T'(w - \delta, w, \delta, n - w, 2\delta).$$

Taking $n_1 = w$, $n_2 = n - w$, $w_1 = w - \delta$, and $w_2 = \delta$ in the above expression completes the proof. ■

Example 12: $T'(1, 5, 5, 13, 10) \leqslant A(18, 10, 6) = 4$ directly by Theorem 27. If, however, Theorem 24 is used as an intermediate step, the bound can be improved to

$$\begin{aligned} T'(1, 5, 5, 13, 10) &\leqslant T'(1, 5, 6, 13, 10) \\ &\leqslant A(19, 10, 6) - 1 = 3. \end{aligned}$$ □

Theorem 28:

$$T'(w_1, n_1, w_2, n_2, d) \leqslant \left\lfloor \frac{n_1}{n_1 - w_1} T'(w_1, n_1 - 1, w_2, n_2, d) \right\rfloor, \quad \text{if } w_1 < n_1$$

$$T'(w_1, n_1, w_2, n_2, d) \leqslant \left\lfloor \frac{n_2}{w_2} T'(w_1, n_1, w_2 - 1, n_2 - 1, d) \right\rfloor, \quad \text{if } w_2 > 0.$$

Proof: Consider a code $\mathcal{C} \in \Phi(\mathcal{H}'(w_1, n_1, w_2, n_2), d)$ and form a new code $\mathcal{C}_j$ by shortening $\mathcal{C}$ in the jth position, where $1 \leqslant j \leqslant n_1$ (this consists of selecting all codewords for which the jth bit is zero and thereafter deleting the jth bit). The total number of zeros in the heads of all codewords of $\mathcal{C}$ equals $\sum_{j=1}^{n_1} |\mathcal{C}_j|$. On the other hand, the same number is lower-bounded by $(n_1 - w_1)|\mathcal{C}|$. Since

$$\mathcal{C}_j \in \Phi(\mathcal{H}'(w_1, n_1 - 1, w_2, n_2), d), \qquad \text{for all } j$$

we have

$$(n_1 - w_1)|\mathcal{C}| \leqslant n_1 T'(w_1, n_1 - 1, w_2, n_2, d).$$

This proves the first inequality in Theorem 28. Similarly, the second inequality is proved by counting in two ways the number of ones in the tails. ■

The next bound is similar to a bound for doubly-constant-weight codes, given by both Levenshtein [39] and Johnson [37, eq. (20)]. We use the notation of [39], which shows the connection with (28).

Theorem 29: If $b > 0$ and $w_1/n_1 \leqslant w_2/n_2$, then

$$T'(w_1, n_1, w_2, n_2, 2\delta) \leqslant \left\lfloor \frac{\delta}{b} \right\rfloor \tag{113}$$

where

$$\begin{aligned} b = \delta &- \frac{w_1(n_1-w_1)}{n_1} - \frac{w_2(n_2-w_2)}{n_2} \\ &+ \frac{n_1}{M^2}\left\{M\frac{w_1}{n_1}\right\}\left\{M\frac{n_1-w_1}{n_1}\right\} \\ &+ \frac{n_2}{M^2}\left\{M\frac{w_2}{n_2}\right\}\left\{M\frac{n_2-w_2}{n_2}\right\} \\ M =& T'(w_1, n_1, w_2, n_2, 2\delta) \end{aligned}$$

and $\{x\}$ denotes the fractional part $x - \lfloor x \rfloor$, as in (41).

Proof: The proof is based upon Proposition 7. We take $n = n_1 + n_2$ and let $\mathcal{C} \in \Phi(\mathcal{H}'(w_1, n_1, w_2, n_2), 2\delta)$. Then the following constraints hold for $f_1, \ldots, f_{n_1+n_2}$:

$$0 \leqslant f_i \leqslant 1, \qquad \text{for } i = 1, \ldots, n_1 + n_2 \tag{114}$$

$$Mf_i \in \mathbb{Z}, \qquad \text{for } i = 1, \ldots, n_1 + n_2 \tag{115}$$

$$\sum_{i=1}^{n_1} f_i \leqslant w_1 \tag{116}$$

$$\sum_{i=1}^{n_1+n_2} f_i = w_1 + w_2. \tag{117}$$

The maximum of $\sum_{i=1}^{n_1+n_2} f_i(1-f_i)$ subject to the constraints (114)–(117) is

$$\begin{aligned} &\frac{w_1(n_1-w_1)}{n_1} + \frac{w_2(n_2-w_2)}{n_2} \\ &\quad - \frac{n_1}{M^2}\left\{M\frac{w_1}{n_1}\right\}\left\{M\frac{n_1-w_1}{n_1}\right\} \\ &\quad - \frac{n_2}{M^2}\left\{M\frac{w_2}{n_2}\right\}\left\{M\frac{n_2-w_2}{n_2}\right\} \end{aligned} \tag{118}$$

if $w_1/n_1 \leqslant w_2/n_2$, and

$$\begin{aligned} &\frac{(w_1+w_2)(n_1+n_2-w_1-w_2)}{n_1+n_2} \\ &- \frac{n_1+n_2}{M^2}\left\{M\frac{w_1+w_2}{n_1+n_2}\right\}\left\{M\frac{n_1+n_2-w_1-w_2}{n_1+n_2}\right\} \end{aligned} \tag{119}$$

otherwise. Substituting (118) for the sum in (32) completes the proof. ■

Remark: An alternative bound is obtained if (119) is substituted for the sum in (32), but this bound has already been covered by a combination of Theorems 10 and 27.

Example 13: From Example 14 in the next subsection, we have $T'(1, 11, 10, 16, 10) \leqslant 14$. Suppose that equality holds. Then Theorem 29 yields $M \leqslant 13$, a contradiction. Hence, $T'(1, 11, 10, 16, 10) \leqslant 13$. Similarly, Theorem 29 reduces the upper bound for $T'(6, 11, 4, 5, 8)$ from 8 (Example 14) to 7. This example continues in Examples 17 and 8. □

B. Binary Doubly-Bounded-Weight Codes as Zonal Codes

In Section III-A, bounds on unrestricted binary codes, constant-weight codes, and doubly-constant-weight codes were obtained by mapping these codes into Euclidean space and applying known bounds for spherical codes. Now, an analogous bound will be derived for doubly-bounded-weight codes. We have found this bound to be particularly successful in conjunction with Proposition 17.

The new bound depends on the existence of upper bounds on the cardinality of zonal codes. One such bound for zonal codes will be presented in the next subsection.

Theorem 30:

$$\begin{aligned} &T'(w_1, n_1, w_2, n_2, d) \\ &\quad \leqslant A_Z(n_1+n_2-1, 1-2d/r^2, \gamma_L, \gamma_H), && \text{if } d \leqslant r^2 \\ &T'(w_1, n_1, w_2, n_2, d) = 1, && \text{if } d > r^2 \end{aligned}$$

where

$$\gamma_L = \arcsin\left(\frac{2c}{r}(n_1w_2 - n_2w_1)\right)$$

$$\gamma_H = \arcsin\left(\frac{2c}{r}(n_1w_2 - n_2w_1 + pn_1 + pn_2)\right)$$

$$c \stackrel{\text{def}}{=} \frac{1}{\sqrt{n_1n_2(n_1+n_2)}} \tag{120}$$

$$r \stackrel{\text{def}}{=} 2\sqrt{\frac{(w_1+w_2)(n_1+n_2-w_1-w_2)}{n_1+n_2}} \tag{121}$$

and p is as defined in (97).

Proof: Let $n = n_1 + n_2$ and $w = w_1 + w_2$. Then $\Omega(\mathcal{H}(n, w))$ is a subset of the $(n-1)$-dimensional sphere, whose radius r_1 and center $\boldsymbol{c}_1$ are given by (19) and (20). Every codeword $\boldsymbol{x}$ of a doubly-bounded-weight code belongs to $\mathcal{H}(n, w)$ and, in addition, satisfies a constraint on $\boldsymbol{x} \cdot \boldsymbol{u}_1$ given in (3). To translate this constraint into a constraint in Euclidean space, we first define a normalized "north pole" vector $\boldsymbol{e}$ in the $(n-1)$-dimensional subspace that contains $\Omega(\mathcal{H}(n, w))$. A vector $\boldsymbol{v} \in \mathbb{R}^n$ belongs to this subspace if and only if $(\boldsymbol{v} - \boldsymbol{c}_1) \cdot \mathbf{1} = 0$. Thus we take

$$\boldsymbol{e} \stackrel{\text{def}}{=} cn\boldsymbol{u}_1 - cn_1\mathbf{1}$$

where $\boldsymbol{u}_1$ is given by (4) and c is given by (120). Notice that $(\boldsymbol{e} - \boldsymbol{c}_1) \cdot \mathbf{1} = 0$ and the constant c in (120) is chosen so that $\|\boldsymbol{e}\|^2 = 1$. Now, from (17) and (97), it follows that any $\boldsymbol{x} \in \mathcal{H}'(w_1, n_1, w_2, n_2)$ satisfies

$$n_1 - 2w_1 \leqslant \Omega(\boldsymbol{x}) \cdot \boldsymbol{u}_1 \leqslant n_1 - 2w_1 + 2p \tag{122}$$

and (16) shows that

$$\Omega(\boldsymbol{x}) \cdot \mathbf{1} = n - 2w = n_1 + n_2 - 2w_1 - 2w_2.$$

We create the Euclidean code

$$\mathcal{C} \stackrel{\text{def}}{=} \left\{\frac{\Omega(\boldsymbol{x}) - \boldsymbol{c}_1}{r} : \boldsymbol{x} \in \mathcal{H}'(w_1, n_1, w_2, n_2)\right\} \tag{123}$$

where $r = r_1$ is given by (121). It is obvious from the normalization in (123) that $\|\boldsymbol{y}\|^2 = 1$ and $\boldsymbol{y} \cdot \mathbf{1} = 0$ for all $\boldsymbol{y} \in \mathcal{C}$, and the fact that

$$\sin \gamma_L \leqslant \boldsymbol{y} \cdot \boldsymbol{e} \leqslant \sin \gamma_H$$

follows from (122). This proves that $\mathcal{C}$, and every subset thereof, is a zonal code. To complete the proof, the maximum cosine is obtained from (10) with $d_E = 2\sqrt{d}$. ■

Example 14: It follows from Theorem 31 (as shown in Example 15) that

$$A_Z(26,\, 41/176,\, \arcsin(47/88),\, \arcsin(11/16)) = 14.$$

Thus Theorem 30 implies that $T'(1,\, 11,\, 10,\, 16,\, 10) \leqslant 14$. Similarly, Theorems 30 and 31 yield $T'(6,\, 11,\, 4,\, 5,\, 8) \leqslant 8$. This example continues in Example 13. □

C. A Bound on Zonal Codes

In this subsection, an upper bound on the cardinality of zonal codes is presented. The proof is deferred to Appendix B. The principal application of this bound is in conjunction with Theorem 30.

Theorem 31: If $0 < \gamma_L \leqslant \gamma_H \leqslant \pi/2$, then

$$A_Z(n, s, \gamma_L, \gamma_H) \leqslant F, \qquad \text{if } \gamma_H > \gamma_G \tag{124}$$
$$A_Z(n, s, \gamma_L, \gamma_H) = 1, \qquad \text{if } s < -\cos 2\gamma_L \tag{125}$$
$$A_Z(n, s, \gamma_L, \gamma_H) = 1 + L, \qquad \text{if } s = \sin \gamma_L,\ \gamma_H = \pi/2 \tag{126}$$
$$A_Z(n, s, \gamma_L, \gamma_H) = L, \qquad \text{otherwise} \tag{127}$$

where

$$F \stackrel{\text{def}}{=} \min \left\{ A_Z(n, s, \gamma_G, \gamma_H) + L,\ A_S\left(n-1,\, \frac{s - \sin\gamma_L \sin\gamma_H}{\cos\gamma_L \cos\gamma_H}\right) \right\}, \qquad \text{if } s < \cos(\gamma_H - \gamma_L)$$
$$F \stackrel{\text{def}}{=} A_Z(n, s, \gamma_G, \gamma_H) + L, \qquad \text{if } s \geqslant \cos(\gamma_H - \gamma_L)$$
$$\gamma_G \stackrel{\text{def}}{=} \pi + \gamma_L - 2 \arctan \frac{s \cot \gamma_L}{1 - s} \tag{128}$$
$$L \stackrel{\text{def}}{=} A_S\left(n-1,\, \frac{s - \sin^2\gamma_L}{\cos^2\gamma_L}\right). \tag{129}$$

Although F in (124) depends on the value of $A_Z(\cdot)$, the foregoing theorem yields a finite bound on $A_Z(n,\, s,\, \gamma_L,\, \gamma_H)$ for any $0 < \gamma_L \leqslant \gamma_H \leqslant \pi/2$ and $-1 \leqslant s < 1$. Typically, case (124) would be applied recursively, each time increasing γ_G, until one of the other cases holds.

Example 15: Consider $n = 26$ and $s = 41/176$. Then for $\gamma_L = \arcsin(47/88)$ and $\gamma_H = \arcsin(11/16)$, we obtain

$$\gamma_G = \pi - \arcsin\left(\tfrac{1363}{4172}\right) > \gamma_H.$$

Since none of (124)–(126) is applicable, we conclude that (127) must hold. Thus

$$A_Z\left(26,\, \tfrac{41}{176},\, \arcsin\left(\tfrac{47}{88}\right),\, \arcsin\left(\tfrac{11}{16}\right)\right) = L = A_S\left(25,\, \tfrac{-3}{41}\right)$$

which, from Example 1, is equal to 14. This example continues in Example 14. □

We point out that the bound of Theorem 31 depends on $A_S(n,\, s)$, the maximum possible cardinality of a spherical code $\mathcal{C} \in \Phi_E(\mathcal{S}(n),\, \sqrt{2-2s})$. For $s \leqslant 0$, the value of $A_S(n,\, s)$ is known exactly (see (11)–(13)) and this is the case where we have found Theorem 31 to be most useful; through Theorem 30 and one of the paths in Fig. 1, numerous upper bounds on $A(n,\, d,\, w)$ were improved. For $s > 0$, we have used Levenshtein's upper bound [40], which resulted in some additional improvements for $T'(w_1,\, n_1,\, w_2,\, n_2,\, d)$ at the expense of higher complexity. However, these improvements did not propagate to $A(n,\, d,\, w)$ or $T(w_1,\, n_1,\, w_2,\, n_2,\, d)$, for $n = n_1 + n_2 \leqslant 28$.

VI. Bounds on $T(w_1,\, n_1,\, w_2,\, n_2,\, d)$

Doubly-constant-weight codes were introduced by Johnson [37] and, independently, by Levenshtein [39] in the early 1970s. Both Johnson [37] and Levenshtein [39] used these codes as a tool to obtain sharper bounds for constant-weight codes, although the specific methods derived in [37] and [39] differ from each other. Best, Brouwer, MacWilliams, Odlyzko, and Sloane [8] gave a linear programming bound for doubly-constant-weight codes. They also applied this and other bounds for doubly-constant-weight codes to sharpen the linear programming bound for constant-weight codes (cf. Proposition 15).

In this section we list all known bounds on doubly-constant-weight codes, including several new ones. Another useful bound is given in Section III-B as Corollary 6.

A. Elementary Bounds

As for $A(n,\, d,\, w)$ and $T'(w_1,\, n_1,\, w_2,\, n_2,\, d)$, we begin the exposition of bounds for doubly-constant-weight codes with some straightforward equalities, given without proof.

Theorem 32:

$$T(w_1, n_1, w_2, n_2, d) = T(w_2, n_2, w_1, n_1, d)$$
$$T(w_1, n_1, w_2, n_2, d) = T(n_1 - w_1, n_1, w_2, n_2, d)$$
$$T(0, n_1, w_2, n_2, d) = A(n_2, d, w_2)$$
$$T(w_1, n_1, 0, n_2, d) = A(n_1, d, w_1)$$
$$T(w_1, n_1, w_2, n_2, 2) = \binom{n_1}{w_1}\binom{n_2}{w_2}$$
$$T(w_1, n_1, w_2, n_2, 2w_1 + 2w_2) = \left\lfloor \frac{n_1}{w_1} \right\rfloor, \qquad \text{if } \frac{w_2}{n_2} \leqslant \frac{w_1}{n_1}$$
$$T(w_1, n_1, w_2, n_2, 2w_1 + 2w_2) = \left\lfloor \frac{n_2}{w_2} \right\rfloor, \qquad \text{if } \frac{w_2}{n_2} > \frac{w_1}{n_1}$$
$$T(w_1, n_1, w_2, n_2, d) = T(w_1, n_1, w_2, n_2, d+1), \qquad \text{if } d \text{ is odd}$$
$$T(w_1, n_1, w_2, n_2, d) = 1, \qquad \text{if } d > 2w_1 + 2w_2.$$

The first two equalities in Theorem 32 are the two basic "reflection operations" for doubly-constant-weight codes. Alternating these operations generates an eightfold symmetry in the T domain, and thereby partitions this domain into eight octants. Thus for all sets of parameters $(w_1,\, n_1,\, w_2,\, n_2,\, d)$, there exists another set that belongs to a given octant and has the same T value. For the sake of brevity, all the theorems in this section are given only for parameters within the octant where $n_1 \leqslant n_2$, $w_1 \leqslant n_1/2$, and $w_2 \leqslant n_2/2$.

Example 16: From Theorem 32, we have

$$\begin{aligned} T(11, 11, 11, 16, 10) &= T(0, 11, 11, 16, 10) \\ &= A(16, 10, 11). \end{aligned}$$

Recall that $A(16, 10, 11) = 3$, as was shown in Example 5. This example continues in Example 8. □

The following theorem consists of four inequalities, all of which can potentially improve upon an upper bound for doubly-constant-weight codes. Hence, all four inequalities should be considered, even when the parameters are confined to one octant only.

Theorem 33:

$$T(w_1,n_1,w_2,n_2,d) \leqslant T'(w_1,n_1,w_2,n_2,d) \quad (130)$$
$$T(w_1,n_1,w_2,n_2,d) \leqslant T'(n_1-w_1,n_1,w_2,n_2,d) \quad (131)$$
$$T(w_1,n_1,w_2,n_2,d) \leqslant T'(w_1,n_1,n_2-w_2,n_2,d) \quad (132)$$
$$T(w_1,n_1,w_2,n_2,d) \leqslant T'(n_1-w_1,n_1,n_2-w_2,n_2,d) \quad (133)$$

Proof: $\mathcal{H}(w_1, n_1, w_2, n_2) \subseteq \mathcal{H}'(w_1, n_1, w_2, n_2)$. ■

Example 17: We have

$$T(10, 11, 10, 16, 10) \leqslant T'(1, 11, 10, 16, 10) \leqslant 13$$

where the last inequality comes from Example 13. This example continues in Example 8. □

Example 18: Combining (130) with (110) yields

$$T(w_1,n_1,w_2,n_2,d) \leqslant A(n_1+n_2,d,w_1+w_2).$$

Of course, this is also immediately clear from the definition of $T(w_1, n_1, w_2, n_2, d)$. This trivial bound, which was known to Levenshtein [39] in 1971, nevertheless updates some of the best known specific upper bounds for doubly-constant-weight codes. For example, $T(2, 6, 5, 15, 10) \leqslant 13$, an improvement from 15 in [8]. □

In analogy with (111), the inequalities in Theorem 33 can be improved upon in some cases, which is our next theorem.

Theorem 34:

$$\begin{aligned} &T(w_1,n_1,w_2,n_2,2\delta) \\ &\quad \leqslant T'(w_1,n_1,w_2,n_2,2\min\{\delta,w_1,n_2-w_2\})-1 \end{aligned} \quad (134)$$
$$\begin{aligned} &T(w_1,n_1,w_2,n_2,2\delta) \\ &\quad \leqslant T'(n_1-w_1,n_1,w_2,n_2,2\min\{\delta,n_1-w_1,n_2-w_2\})-1 \end{aligned} \quad (135)$$
$$\begin{aligned} &T(w_1,n_1,w_2,n_2,2\delta) \\ &\quad \leqslant T'(w_1,n_1,n_2-w_2,n_2,2\min\{\delta,w_1,w_2\})-1 \end{aligned} \quad (136)$$
$$\begin{aligned} &T(w_1,n_1,w_2,n_2,2\delta) \\ &\quad \leqslant T'(n_1-w_1,n_1,n_2-w_2,n_2,2\min\{\delta,n_1-w_1,w_2\})-1. \end{aligned} \quad (137)$$

Proof: Consider a code $\mathcal{C} \in \Phi(\mathcal{H}(w_1, n_1, w_2, n_2), 2\delta)$ and define $\mathcal{C}' \stackrel{\text{def}}{=} \mathcal{C} \cup \{\boldsymbol{c}'\}$, where

$$\boldsymbol{c}' = (\underbrace{0, \ldots, 0}_{n_1+n_2-w_1-w_2}, \underbrace{1, \ldots, 1}_{w_1+w_2}). \quad (138)$$

There are two cases, depending on whether $w_1 + w_2 \leqslant n_2$ or not. It is easily verified that $\mathcal{C}' \in \mathcal{H}'(w_1, n_1, w_2, n_2)$ in both cases. The minimum distance of $\mathcal{C}'$ is given by (112), where

$$d(\boldsymbol{c}, \boldsymbol{c}') \geqslant 2w_1, \qquad \text{if } w_1+w_2 \leqslant n_2 \quad (139)$$
$$d(\boldsymbol{c}, \boldsymbol{c}') \geqslant 2n_2 - 2w_2, \qquad \text{if } w_1+w_2 \geqslant n_2 \quad (140)$$

or, equivalently,

$$d(\boldsymbol{c}, \boldsymbol{c}') \geqslant 2\min\{w_1, n_2-w_2\} \quad (141)$$

for all $\boldsymbol{c} \in \mathcal{C}'$, which completes the proof of (134). The bounds (135)–(137) follow from repeated application of the first two equalities in Theorem 32. ■

The following theorem is due to Levenshtein [39]. Note that the right-hand sides are independent of n_1 and n_2, respectively.

Theorem 35:

$$T(w_1,n_1,w_2,n_2,d) \leqslant A(n_2, d-2w_1, w_2), \qquad \text{if } d > 2w_1$$
$$T(w_1,n_1,w_2,n_2,d) \leqslant A(n_1, d-2w_2, w_1), \qquad \text{if } d > 2w_2$$

The following bounds, analogous to Theorems 9 and 28, were first given by Johnson [37].

Theorem 36:

$$T(w_1,n_1,w_2,n_2,d) \leqslant \left\lfloor \frac{n_1}{w_1} T(w_1-1, n_1-1, w_2, n_2, d) \right\rfloor, \quad \text{if } w_1 > 0 \quad (142)$$
$$T(w_1,n_1,w_2,n_2,d) \leqslant \left\lfloor \frac{n_1}{n_1-w_1} T(w_1, n_1-1, w_2, n_2, d) \right\rfloor, \quad \text{if } w_1 < n_1 \quad (143)$$
$$T(w_1,n_1,w_2,n_2,d) \leqslant \left\lfloor \frac{n_2}{w_2} T(w_1, n_1, w_2-1, n_2-1, d) \right\rfloor, \quad \text{if } w_2 > 0 \quad (144)$$
$$T(w_1,n_1,w_2,n_2,d) \leqslant \left\lfloor \frac{n_2}{n_2-w_2} T(w_1, n_1, w_2, n_2-1, d) \right\rfloor, \quad \text{if } w_2 < n_2. \quad (145)$$

Remark: Bounds analogous to (142) and (145) do not exist for doubly-bounded-weight codes, since the number of ones in the heads and the number of zeros in the tails are not lower-bounded in this case.

B. Linear Programming

A distance distribution can be defined for doubly-constant-weight codes, whose components are indexed by two variables. We refer the reader to [8] for more details. Based on this distribution, the following linear programming bound was given in [8].

Theorem 37:

$$T(w_1, n_1, w_2, n_2, 2\delta) \leqslant 1 + \left\lfloor \max \sum_{i=i_0}^{w_1} \sum_{j=j_0}^{w_2} A_{2i,2j} \right\rfloor$$

where $i_0 = \max\{0, \delta - w_2\}$ and $j_0 = \max\{0, \delta - i\}$. The set of optimization variables consists of all $A_{2i,2j}$ for which $0 \leqslant i \leqslant w_1$, $0 \leqslant j \leqslant w_2$, and $i+j \geqslant \delta$, while the maximization is carried out over all sets of these variables that satisfy $A_{2i,2j} \geqslant 0$ and Proposition 38.

The main set of constraints for this linear programming bound is given by the following proposition [8].

Proposition 38: For all $k = 0, \ldots, w_1$ and for all $l = 0, \ldots, w_2$

$$\sum_{i=i_0}^{w_1} q(k, i, n_1, w_1) \sum_{j=j_0}^{w_2} q(l, j, n_2, w_2) A_{2i,2j} \geqslant -1$$

where q is defined by (56) and i_0, j_0 are as in Theorem 37.

C. Specific Bounds

To the best of our knowledge, the only specific upper bound for doubly-constant-weight codes has been reported in [31], namely, $T(1, 6, 6, 15, 10) \leqslant 7$. This was later identified as a typographical error in [17].

D. Redundant Bounds

We now list bounds on doubly-constant-weight codes that were evaluated but did not yield any competitive values within the studied range of parameters.

The bounds [39, eq. (8)] and [37, eq. (19)], which despite disparate notation are completely equivalent, are inferior to Corollary 6. The bounds [39, eq. (11)] and [37, eq. (20)] are also equivalent to each other, and they are precisely what one gets by combining Theorems 29 and 33.

Theorem 3 is a strong bound, but only when $s \leqslant 0$. This special case is Corollary 6. When $s > 0$, Theorem 3 can be evaluated using the bound of Levenshtein [40] for $A(n, s)$. This, however, does not improve upon the values obtained through Theorems 32–37 within the studied range of parameters.

VII. The Tables

This section contains tables of the best known bounds on $A(n, d, w)$, which were obtained using the results presented in this paper. The authors would appreciate hearing of any improvements to the tables. To conserve space, our tables of upper bounds for $T'(w_1, n_1, w_2, n_2, d)$ and $T(w_1, n_1, w_2, n_2, d)$ are published electronically only [3]. On the same website [3], we will also keep record of any updates or corrections that are brought to our attention.

Most of the theorems in this paper yield upper bounds that depend on $A(n, d, w)$, $T'(w_1, n_1, w_2, n_2, d)$, or $T(w_1, n_1, w_2, n_2, d)$. However, these entities are in general not known exactly. This problem is easily overcome by substituting any upper bound for the exact value. This strategy of obtaining upper bounds based on other bounds yields a complicated pattern of dependencies, as shown in Fig. 1. To provide each theorem with the best possible input, the loops in this figure were evaluated iteratively until a steady state was reached.

The tables also reference the number of the theorem from which each bound was obtained. Although, in many cases, the same bound can be obtained using more than one method, we mention only one method for each bound. In this regard, we have given precedence to universal methods (as opposed to methods applicable to certain parameters only), to analytical methods (as opposed to computerized search methods), and to relatively simple methods. We have also tried to keep the total number of methods used in the production of the tables at a minimum.

Tables I–VI give upper and lower bounds on $A(n, d, w)$ for all $n \leqslant 28$ and all even $d \leqslant 14$. For each n and d, w ranges from $d/2+1$ to $\lfloor n/2 \rfloor$. The values of $A(n, d, w)$ for w outside this interval or for odd d are given by Theorem 8. Finally, for $n \leqslant 28$ and $d = 16$ or 18, exact values of $A(n, d, w)$ are given in [17].

All the lower bounds in Tables I–VI are taken from http://www.research.att.com/~njas/codes/Andw/, an updated and extended version of [17]. Boldface indicates updates to the upper bounds in the tables of [34] and http://www.research.att.com/~njas/codes/Andw/. Those tables cover $n \leqslant 24$ for $d \leqslant 10$ and $n \leqslant 27$ for $d = 12$. Superscripts refer to theorem numbers in this paper.

One can conclude that most progress since similar tables were last published has been made for $d \geqslant 8$. Out of the 23 unresolved instances for $d = 8$ in [17], [34] fourteen have now been updated. For $d = 10$, ten out of eighteen instances are updated, of which two are settled exactly. The corresponding numbers for $d = 12$ and $d = 14$ are, respectively, six out of 13 with three exact values and three out of three with two exact values.

Appendix A
Errata in Earlier Work

As pointed out in [37], there exist errors in some of the published literature on constant-weight codes. Johnson [37] provides a list of known errata. A similar but more extensive list, covering more recent literature, was included in [17]. In this section, we supplement these two lists with many newly discovered errata, and also comment on some of the known ones. We do not, however, list *all* errata previously reported.

The bounds $A(9, 4) \leqslant 20$, $A(10, 4) \leqslant 39$, $A(11, 4) \leqslant 82$, and $A(12, 4) \leqslant 154$, which were claimed by Wax [55], cannot be obtained by the methods proposed in [55]. This was proved in [8]. In fact, no useful contributions remain today from the Wax [55] bound.

Johnson [35] claimed without proof that $A(15,6) \leqslant 127$, $A(16,6) \leqslant 248$, $A(14,6,5) \leqslant 27$, and $A(16,6,5) \leqslant 40$. These are incorrect, as these bounds do not agree with the exact values that are well known today [42, pp. 674, 686].

The following corrections relate to the well-known paper of Best, Brouwer, MacWilliams, Odlyzko, and Sloane [8]. In [8, legend of Table IIA], "[e]From Theorem 9 …" and "[f]From The-

TABLE I
BOUNDS ON $A(n, 4, w)$

n	$w=3$	4	5	6	7	8	9
6	4[5]						
7	7[5]						
8	8[5]	14[5]					
9	12[9]	18[9]					
10	13[9]	30[9]	36[9]				
11	17[9]	35[9]	66[9]				
12	20[9]	51[9]	80 – 84[9]	132[9]			
13	26[9]	65[9]	123 – 132[9]	166 – 182[9]			
14	28[9]	91[9]	169 – 182[9]	278 – 308[9]	325 – 364[9]		
15	35[9]	105[9]	237 – 271[13]	389 – 455[9]	585 – 660[9]		
16	37[9]	140[9]	315 – 336[9]	615 – 722[9]	836 – 1040[9]	1170 – 1320[9]	
17	44[9]	156 – 157[9]	441 – 476[9]	854 – 952[9]	1416 – 1753[9]	1770 – 2210[9]	
18	48[9]	198[9]	518 – 565[9]	1260 – 1428[9]	2041 – 2448[9]	3186 – 3944[9]	3540 – 4420[9]
19	57[9]	228[9]	692 – 752[9]	1620 – 1789[9]	3172 – 3876[9]	4667 – 5814[9]	6726 – 8326[9]
20	60[9]	285[9]	874 – 912[9]	2304 – 2506[9]	4213 – 5111[9]	7730 – 9690[9]	10039 – 12920[9]
21	70[9]	315[9]	1071 – 1197[9]	2856 – 3192[9]	6156 – 7518[9]	10753 – 13416[9]	16897 – 22610[9]
22	73[9]	385[9]	1386[9]	3927 – 4389[9]	8252 – 10032[9]	16430 – 20674[9]	25570 – 32794[9]
23	83[9]	418 – 419[9]	1771[9]	5313[9]	11638 – 14421[9]	23276 – 28842[9]	40786 – 52833[9]
24	88[9]	498[9]	1895 – 2011[9]	7084[9]	15656 – 18216[9]	34914 – 43263[9]	59387 – 76912[9]
25	100[9]	550[9]	2334 – **2490**[9]	7772 – **8379**[9]	21106 – **25300**[9]	46872 – **56925**[9]	88748 – **120175**[9]
26	104[9]	650[9]	2670 – **2860**[9]	10010 – **10790**[9]	26920 – **31122**[9]	65364 – **82225**[9]	128050 – **164450**[9]
27	117[9]	702[9]	3276 – **3510**[9]	12012 – **12870**[9]	35510 – **41618**[9]	87709 – **105036**[9]	186058 – **246675**[9]
28	121[9]	819[9]	3718 – **3931**[9]	15288 – **16380**[9]	44747 – **51480**[9]	121403 – **145663**[9]	260224 – **326778**[9]

n	$w=10$	11	12	13	14
20	13452 – 16652[9]				
21	20188 – 27132[9]				
22	36381 – 49742[9]	39688 – 54264[9]			
23	57436 – 75426[9]	73794 – 104006[9]			
24	96496 – 126799[9]	116937 – 164565[9]	146552 – 208012[9]		
25	140605 – **192280**[9]	196449 – **288179**[9]	228901 – **342843**[9]		
26	218905 – **312455**[9]	315700 – **454480**[9]	398381 – **624387**[9]	425950 – **685686**[9]	
27	330347 – **444015**[9]	510571 – **766935**[9]	675262 – **1022580**[9]	778872 – **1296803**[9]	
28	502068 – **690690**[9]	806303 – **1130220**[9]	1154541 – **1789515**[9]	1400118 – **2202480**[9]	1520224 – **2593606**[9]

orem 6 …” should both be replaced by a reference to the unnamed theorem immediately before [8, Sec. IV-A]. In the same legend to [8, Table IIA], the reference “[q]See [31], [34]” does not apply for $A(12, 6, 5)$ and $A(13, 6, 5)$; see Example 3 and Theorem 21 in the present paper. To quote [17], all the linear programming bounds for $d = 10$ in [8, Table IID] should “be regarded with suspicion” until further checks are made. Our checks and Honkala's [34] together verify all of these bounds. There are three more errors in [8, Table III], in addition to the five errors reported in [17]. The bounds $T(2,5,7,16,10) \leqslant 30$ and $T(3, 6, 7, 16, 10) \leqslant 60$ originate from the known error $T(2, 4, 7, 16, 10) \leqslant 18$, which was corrected in [17]. Our best upper bounds in these cases are $T(2, 5, 7, 16, 10) \leqslant 31$ and $T(3, 6, 7, 16, 10) \leqslant 62$. In [8, Table IIIC], the value of $T(3, 8, 3, 7, 10)$ should be 3, not 2. Also, in the last two lines of [8, p. 85], “B_i” should be “A_i,” while “$\lambda\binom{w}{2}$” in [8, Theorem 20] should be “$\lambda\binom{M}{2}$.”

In [42, p. 689], the values of $A(16, 10, 7)$ and $A(16, 10, 9)$ should be 4, not 3. The linear programming bounds for $d = 10$ are as unreliable in [42] as in [8]; see above.

The foregoing comments on [42] apply to [31] as well. In addition, “[13, (29)]” in [31, p. 40, line 32] should be “[13, (27)]” and “[5, Table IIIA]” three lines later should be “[3, Table IIIA].”

Furthermore, in [17, Table III], “$A(23, 10, 7) = 21$” should be “$A(23, 10, 7) = 20$” and the corresponding entries in [17, Tables I-D and XVI] should give 20 as an exact value [49]. The value $A(21, 10, 8) = 21$ in [17, Table I-D] is not explained in [17, Table III]. It appears possible that [17, Table I-D] was wrong in stating that the value for $A(21, 10, 8)$ was exact rather than a lower bound [49]. Also, $T(2, 4, 7, 16, 10) \geqslant 19$ [17, p. 1359, line 11] should be $T(2, 4, 7, 16, 10) \leqslant 19$ and “line 3” [17, p. 1360, line 13] should be “line 23.”

Finally, in [1, eq. (3)], “$\leqslant$” should be “$\geqslant$.”

As demonstrated by this list of errata, and by the lists in [37] and [17], it is very difficult to collect a large number of bounds without introducing some errors. We would welcome reports of any corrections and updates to this work.

APPENDIX B
PROOF OF THEOREM 31

In this appendix, we prove the bound on the cardinality of zonal codes given as Theorem 31 in Section V-C. We distinguish between two cases: $\gamma_H < \pi/2$ and $\gamma_H = \pi/2$. Upper bounds for these two cases will be derived separately in Lemmas 43 and 44, respectively. These two lemmas, along with the lower bound of Lemma 45, yield Theorem 31.

TABLE II
BOUNDS ON $A(n, 6, w)$

n	$w=4$	5	6	7	8	9
8	2^{10}					
9	3^{5}					
10	5^{5}	6^{5}				
11	6^{5}	11^{5}				
12	9^{5}	12^{5}	22^{5}			
13	13^{5}	18^{21}	26^{9}			
14	14^{5}	28^{20}	42^{20}	42^{21}		
15	15^{5}	42^{9}	70^{20}	$69-\mathbf{78}^{9}$		
16	20^{9}	48^{9}	112^{9}	$109-\mathbf{138}^{9}$	$120-150^{20}$	
17	20^{21}	68^{9}	$112-136^{9}$	$166-\mathbf{234}^{9}$	$184-283^{20}$	
18	22^{9}	$69-72^{9}$	$132-202^{13}$	$243-349^{9}$	$260-428^{20}$	$304-425^{20}$
19	25^{21}	$76-83^{9}$	$172-228^{9}$	$338-520^{20}$	$408-734^{14}$	$504-789^{20}$
20	30^{9}	$84-100^{9}$	$232-276^{9}$	$462-651^{9}$	$588-1107^{14}$	$832-1363^{20}$
21	31^{9}	$108-126^{9}$	$269-350^{9}$	$570-828^{9}$	$774-1695^{14}$	$1184-2364^{20}$
22	37^{9}	$132-136^{9}$	$319-462^{9}$	$759-1100^{9}$	$1139-2277^{9}$	$1792-3775^{20}$
23	40^{9}	$147-170^{9}$	$399-521^{9}$	$969-1518^{9}$	$1436-3162^{9}$	$2271-5819^{9}$
24	42^{9}	$168-192^{9}$	$532-680^{9}$	$1368-1786^{9}$	$1882-4554^{9}$	$3041-8432^{9}$
25	50^{9}	210^{9}	$700-\mathbf{800}^{9}$	$1900-\mathbf{2428}^{9}$	$2590-\mathbf{5581}^{9}$	$4127-\mathbf{12620}^{14}$
26	52^{9}	260^{9}	910^{9}	$2600-\mathbf{2971}^{9}$	$3532-\mathbf{7891}^{9}$	$5703-\mathbf{16122}^{9}$
27	54^{9}	$260-\mathbf{280}^{9}$	1170^{9}	3510^{9}	$4786-\mathbf{10027}^{9}$	$7727-\mathbf{23673}^{9}$
28	63^{9}	$280-\mathbf{302}^{9}$	$1170-\mathbf{1306}^{9}$	4680^{9}	$6315-\mathbf{12285}^{9}$	$10313-\mathbf{31195}^{9}$

n	$w=10$	11	12	13	14
20	$944-1421^{20}$				
21	$1454-2702^{20}$				
22	$2182-4416^{20}$	$2636-5064^{20}$			
23	$2970-7521^{20}$	$3585-7953^{20}$			
24	$4200-12186^{14}$	$5267-14682^{9}$	$5616-15906^{20}$		
25	$6036-\mathbf{19037}^{14}$	$7960-\mathbf{24630}^{20}$	$9031-\mathbf{30587}^{9}$		
26	$8695-\mathbf{28893}^{14}$	$12037-\mathbf{42081}^{20}$	$14836-\mathbf{50204}^{20}$	$15977-\mathbf{61174}^{9}$	
27	$12368-\mathbf{43529}^{9}$	$18096-\mathbf{66079}^{20}$	$23879-\mathbf{84574}^{20}$	$27553-\mathbf{91080}^{20}$	
28	$17447-\mathbf{63756}^{14}$	$29484-\mathbf{104231}^{20}$	$40188-\mathbf{142117}^{14}$	$49462-\mathbf{164220}^{20}$	$52995-\mathbf{169740}^{20}$

Throughout this appendix, s denotes the maximum cosine between points of a zonal code, as defined in (10). Thus

$$-1 \leqslant s < 1.$$

We will make use of the function $f(\cdot)$ and the angle γ_G, defined as follows. For any $-\pi/2 < \alpha, \beta < \pi/2$

$$f(s, \alpha, \beta) \stackrel{\text{def}}{=} \frac{s - \sin\alpha \sin\beta}{\cos\alpha\cos\beta} \tag{146}$$

and for any $\gamma_L \in (0, \pi/2)$ the angle[2]

$$\gamma_G \stackrel{\text{def}}{=} \pi + \gamma_L - 2\arctan\frac{s\cot\gamma_L}{1-s}. \tag{147}$$

The angle γ_G was already defined in (128) of Theorem 31. Here, we point out that this definition is motivated by the following property. As will be shown in Lemma 42, for γ_G as defined in (128) and (147), we have

$$f(s, \gamma_L, \gamma_L) = f(s, \gamma_L, \gamma_G).$$

Also note that as s decreases from 1 to $\sin\gamma_L$, the angle γ_G increases monotonically from γ_L to $\pi/2$. The following lemma gives some important bounds on γ_G.

[2] We intentionally avoid the inverse cotangent, since there is no uniform agreement on the definition of $\operatorname{arccot} x$ for $x < 0$.

Lemma 39: If $s < \sin\gamma_L$, then $\gamma_G > \gamma_L + \arccos s > \pi/2$. If $s = \sin\gamma_L$, then $\gamma_G = \gamma_L + \arccos s = \pi/2$. If $s > \sin\gamma_L$, then $\gamma_G < \gamma_L + \arccos s < \pi/2$.

Proof: Follows by rewriting (147) as

$$\gamma_G = \pi + \gamma_L - 2\arctan\left(\frac{\tan\arcsin s}{\tan\gamma_L}\cdot\tan\frac{\arccos(-s)}{2}\right). \quad\blacksquare$$

The next three lemmas will be proved independently of each other, and then combined in Lemma 43. The main idea of the following lemma is that the "latitudes" of points in a zonal code are bounded by a function of γ_L and s, rather than by γ_H, provided s is within a certain range.

Lemma 40: If $-\gamma_H < \gamma_L \leqslant \gamma_H \leqslant \pi/2$, then

$$A_Z(n, s, \gamma_L, \gamma_H) = A_Z(n, s, \gamma_L, \pi - \gamma_L - \arccos s), \quad \text{if } -\cos 2\gamma_L \leqslant s \leqslant -\cos(\gamma_L + \gamma_H) \tag{148}$$

$$A_Z(n, s, \gamma_L, \gamma_H) = 1, \qquad \text{if } s < -\cos 2\gamma_L \tag{149}$$

Proof: Consider a zonal code $\mathcal{C}$ with $|\mathcal{C}| \geqslant 2$, and let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two arbitrary points in $\mathcal{C}$. Now $\boldsymbol{x}$, $\boldsymbol{y}$, and the north pole vector $\boldsymbol{e}$ form a spherical triangle with sides $\arccos \boldsymbol{x}\cdot\boldsymbol{e}$,

TABLE III
BOUNDS ON $A(n,\, 8,\, w)$

n	$w=5$	6	7	8	9	10
10	2^{5}					
11	2^{10}					
12	3^{5}	4^{5}				
13	3^{10}	4^{10}				
14	4^{10}	7^{5}	8^{5}			
15	6^{5}	10^{5}	15^{5}			
16	6^{10}	16^{5}	16^{5}	30^{5}		
17	7^{10}	17^{5}	24^{21}	34^{9}		
18	9^{10}	21^{9}	$33-\mathbf{39}^{9}$	$46-\mathbf{54}^{9}$	$48-\mathbf{68}^{9}$	
19	12^{5}	28^{9}	$52-57^{9}$	$78-\mathbf{92}^{9}$	$88-\mathbf{114}^{9}$	
20	16^{5}	40^{9}	80^{9}	$130-142^{9}$	$160-\mathbf{195}^{20}$	$176-\mathbf{228}^{9}$
21	21^{5}	56^{9}	120^{9}	210^{9}	$280-\mathbf{320}^{20}$	$336-399^{21}$
22	21^{21}	77^{9}	176^{9}	330^{9}	$280-493^{14}$	$616-\mathbf{641}^{20}$
23	23^{5}	$77-80^{9}$	253^{9}	506^{9}	$400-\mathbf{796}^{20}$	$616-\mathbf{1109}^{21}$
24	24^{5}	$78-92^{9}$	$253-274^{9}$	759^{9}	$640-1143^{14}$	$960-1639^{20}$
25	30^{9}	100^{9}	$254-\mathbf{328}^{9}$	$759-\mathbf{856}^{9}$	$829-\mathbf{1610}^{14}$	$1248-\mathbf{2448}^{20}$
26	30^{21}	130^{9}	$257-\mathbf{371}^{9}$	$760-\mathbf{1066}^{9}$	$883-\mathbf{2160}^{14}$	$1519-\mathbf{3719}^{20}$
27	$31-\mathbf{32}^{9}$	$130-\mathbf{135}^{9}$	$278-\mathbf{500}^{9}$	$766-\mathbf{1252}^{9}$	$970-\mathbf{2914}^{14}$	$1597-\mathbf{5260}^{20}$
28	33^{9}	$130-\mathbf{149}^{9}$	$296-\mathbf{540}^{9}$	$833-\mathbf{1750}^{9}$	$1107-\mathbf{3895}^{9}$	$1820-\mathbf{7368}^{20}$

n	$w=11$	12	13	14
22	$672-\mathbf{766}^{20}$			
23	$1288-\mathbf{1328}^{20}$			
24	$1288-\mathbf{2188}^{20}$	2576^{20}		
25	$1662-\mathbf{3575}^{20}$	$2576-\mathbf{4169}^{20}$		
26	$1988-\mathbf{5315}^{20}$	$3070-\mathbf{6834}^{20}$	$3588-\mathbf{7164}^{20}$	
27	$2295-\mathbf{7837}^{20}$	$3335-\mathbf{10547}^{20}$	$4094-\mathbf{11991}^{20}$	
28	$2756-\mathbf{11939}^{14}$	$4916-\mathbf{17299}^{20}$	$4805-\mathbf{21739}^{20}$	$6090-\mathbf{23268}^{20}$

$\arccos \boldsymbol{y}\cdot\boldsymbol{e}$, and $\arccos \boldsymbol{x}\cdot\boldsymbol{y}$. The triangle inequality for spherical triangles [46, p. 75] implies that

$$\begin{aligned}\arccos \boldsymbol{x}\cdot\boldsymbol{e} &\geqslant \arccos \boldsymbol{x}\cdot\boldsymbol{y} - \arccos \boldsymbol{y}\cdot\boldsymbol{e}\\ &\geqslant \arccos s - \left(\frac{\pi}{2} - \gamma_L\right)\end{aligned}$$

or, equivalently,

$$\arcsin \boldsymbol{x}\cdot\boldsymbol{e} \leqslant \pi - \gamma_L - \arccos s. \tag{150}$$

If $s < -\cos 2\gamma_L$, then (150) yields $\boldsymbol{x}\cdot\boldsymbol{e} < \sin\gamma_L$, which is a contradiction. Thus in this case, $\mathcal{C}$ cannot contain two or more points, which proves (149). If $s > -\cos(\gamma_L + \gamma_H)$ or if $\gamma_L + \gamma_H < 0$, then the inequality (150) is weaker than

$$\boldsymbol{x}\cdot\boldsymbol{e} \leqslant \sin\gamma_H \tag{151}$$

which follows directly from the definition of a zonal code. On the other hand, for $\gamma_L > -\gamma_H$ and for s in the range specified in (148), the bound (150) is stronger than (151), which completes the proof of the lemma. ∎

The main idea of the following lemma is the construction of spherical codes from zonal codes. This makes it possible to use bounds for spherical codes in the case of zonal codes.

Lemma 41: For all $-\pi/2 < \gamma_L \leqslant \gamma_H < \pi/2$ and for s in the range $-\cos 2\gamma_L \leqslant s < \cos(\gamma_H - \gamma_L)$, we have

$$A_Z(n, s, \gamma_L, \gamma_H) \leqslant A_S\left(n-1, \max_{\gamma_L \leqslant \alpha, \beta \leqslant \gamma_H} f(s,\alpha,\beta)\right).$$

Proof: Let $\mathcal{C} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_M\}$ be a zonal code, and let $\boldsymbol{e}$ be its north pole vector. For $i = 1, 2, \ldots, M$, we let $\gamma_i = \arcsin \boldsymbol{x}_i\cdot\boldsymbol{e}$ denote the "latitudes" of the points of $\mathcal{C}$. Consider the code $\mathcal{C}_S = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_M\}$, where

$$\boldsymbol{y}_i \stackrel{\text{def}}{=} \frac{\boldsymbol{x}_i - \boldsymbol{e}\sin\gamma_i}{\cos\gamma_i} \tag{152}$$

for $i = 1, 2, \ldots, M$. It is easy to verify that $\|\boldsymbol{y}_i\| = 1$ and $\boldsymbol{y}_i\cdot\boldsymbol{e} = 0$ for all i. Furthermore $\boldsymbol{y}_i\cdot\boldsymbol{y}_j = f(\boldsymbol{x}_i\cdot\boldsymbol{x}_j, \gamma_i, \gamma_j)$ for all distinct $1 \leqslant i, j \leqslant M$. Hence, $\mathcal{C}_S$ is a spherical code in $n-1$ dimensions with a maximum cosine given by

$$\begin{aligned}\max_{i\neq j} \boldsymbol{y}_i\cdot\boldsymbol{y}_j &= \max_{i\neq j} f(\boldsymbol{x}_i\cdot\boldsymbol{x}_j, \gamma_i, \gamma_j)\\ &\leqslant \max_{\gamma_L \leqslant \alpha, \beta \leqslant \gamma_H} f(s, \alpha, \beta).\end{aligned}$$

The constraints on s in the statement of the lemma ensure that $-1 \leqslant \max f(s, \alpha, \beta) < 1$. ∎

The next lemma is concerned with the maximization of the function $f(\cdot)$ defined in (146).

Lemma 42: For all $0 < \gamma_L \leqslant \gamma_H < \pi/2$, we have

$$\begin{aligned}\max_{\gamma_L \leqslant \alpha,\beta \leqslant \gamma_H} f(s, \alpha, \beta) &= f(s, \gamma_L, \gamma_L), &&\text{if } \gamma_H \leqslant \gamma_G\\ \max_{\gamma_L \leqslant \alpha,\beta \leqslant \gamma_H} f(s, \alpha, \beta) &= f(s, \gamma_L, \gamma_H), &&\text{if } \gamma_H \geqslant \gamma_G.\end{aligned}$$

Proof: Regard $f(s, \alpha, \beta)$ as a function of α, keeping s and β fixed. Since $df/d\alpha$ is well-defined, the maximum occurs

TABLE IV
BOUNDS ON $A(n,\, 10,\, w)$

n	w = 6	7	8	9	10	11	12	13	14
12	2^{5}								
13	2^{5}								
14	2^{10}	2^{10}							
15	3^{5}	3^{5}							
16	3^{10}	4^{5}	4^{10}						
17	3^{10}	5^{5}	6^{5}						
18	4^{10}	6^{5}	9^{5}	10^{5}					
19	4^{10}	8^{5}	12^{10}	19^{5}					
20	5^{10}	10^{10}	17^{21}	$\mathbf{20}^{5}$	38^{5}				
21	7^{5}	13^{11}	$\mathbf{21}^{5}$	$27-\mathbf{35}^{9}$	$38-\mathbf{42}^{9}$				
22	7^{5}	16^{21}	$24-\mathbf{33}^{9}$	$35-\mathbf{51}^{9}$	$46-73^{21}$	$46-81^{21}$			
23	8^{5}	20^{21}	$33-\mathbf{46}^{9}$	$45-\mathbf{81}^{20}$	$54-117^{9}$	$65-135^{20}$			
24	9^{10}	$\mathbf{24}^{5}$	$38-\mathbf{60}^{9}$	$56-119^{20}$	$72-171^{20}$	$95-223^{20}$	$122-247^{20}$		
25	10^{10}	$28-\mathbf{32}^{9}$	$48-\mathbf{75}^{9}$	$72-\mathbf{158}^{20}$	$100-\mathbf{262}^{20}$	$125-\mathbf{388}^{20}$	$132-\mathbf{464}^{9}$		
26	13^{5}	$28-\mathbf{36}^{14}$	$54-\mathbf{104}^{9}$	$91-\mathbf{214}^{20}$	$130-\mathbf{410}^{9}$	$168-\mathbf{581}^{20}$	$195-\mathbf{728}^{20}$	$210-\mathbf{869}^{20}$	
27	14^{10}	$36-\mathbf{48}^{14}$	$66-\mathbf{121}^{9}$	$118-\mathbf{299}^{20}$	$162-\mathbf{577}^{9}$	$222-\mathbf{900}^{20}$	$351-\mathbf{1289}^{20}$	$405-\mathbf{1460}^{20}$	
28	16^{10}	$37-\mathbf{56}^{9}$	$78-\mathbf{168}^{9}$	$132-\mathbf{376}^{9}$	$210-\mathbf{821}^{20}$	$286-\mathbf{1434}^{20}$	$365-\mathbf{1981}^{20}$	$756-\mathbf{2438}^{20}$	$790-\mathbf{2629}^{20}$

TABLE V
BOUNDS ON $A(n,\, 12,\, w)$

n	w = 7	8	9	10	11	12	13	14
14	2^{5}							
15	2^{5}							
16	2^{5}	2^{10}						
17	2^{10}	2^{10}						
18	3^{5}	3^{5}	4^{5}					
19	3^{5}	3^{10}	4^{5}					
20	3^{10}	5^{5}	5^{5}	6^{5}				
21	3^{10}	5^{5}	7^{5}	7^{5}				
22	4^{5}	6^{5}	8^{5}	11^{5}	12^{5}			
23	4^{10}	6^{10}	10^{10}	16^{10}	23^{5}			
24	4^{10}	9^{5}	16^{5}	24^{5}	$\mathbf{24}^{5}$	46^{5}		
25	5^{10}	10^{5}	25^{5}	$28-38^{20}$	$36-\mathbf{42}^{9}$	$\mathbf{50}^{9}$		
26	5^{10}	13^{5}	$\mathbf{26}^{5}$	$33-\mathbf{48}^{20}$	$39-69^{21}$	$54-83^{21}$	$58-92^{21}$	
27	6^{10}	15^{10}	39^{9}	$39-65^{21}$	$54-100^{21}$	$82-140^{20}$	$86-\mathbf{156}^{20}$	
28	8^{5}	19^{11}	$39-\mathbf{45}^{20}$	$49-\mathbf{99}^{20}$	$65-\mathbf{149}^{20}$	$84-\mathbf{199}^{20}$	$99-\mathbf{245}^{20}$	$172-\mathbf{265}^{20}$

either at an endpoint of the interval $\gamma_L \leqslant \alpha \leqslant \gamma_H$ or at an interior point α for which $df/d\alpha = 0$ and $d^2 f/d\alpha^2 \leqslant 0$. By differentiating $f(s,\, \alpha,\, \beta)$ twice with respect to α and observing that $0 < \beta < \pi/2$, it is straightforward to verify that the maximum does not occur at an interior point. Hence, $f(s,\, \alpha,\, \beta)$ is maximum for either $\alpha = \gamma_L$ or $\alpha = \gamma_H$. The same argument proves that the maximum occurs for $\beta = \gamma_L$ or $\beta = \gamma_H$. Thus the function $f(s,\, \alpha,\, \beta)$ attains its global maximum at one of the four corners of the feasibility region $\gamma_L \leqslant \alpha,\, \beta \leqslant \gamma_H$ in the $(\alpha,\, \beta)$-plane.

Since $f(s,\, \alpha,\, \beta)$ is a symmetric function of α and β, we have $f(s,\, \gamma_H,\, \gamma_L) = f(s,\, \gamma_L,\, \gamma_H)$. Also, it is obvious that $f(s,\, \gamma_H,\, \gamma_H) \leqslant f(s,\, \gamma_L,\, \gamma_L)$ for all $0 < \gamma_L \leqslant \gamma_H < \pi/2$. Thus it remains to compare $f(s,\, \gamma_L,\, \gamma_L)$ and $f(s,\, \gamma_L,\, \gamma_H)$. We factorize the difference. Omitting the tedious details, the result can be written as

$$f(s,\, \gamma_L,\, \gamma_L) - f(s,\, \gamma_L,\, \gamma_H) = \frac{1-\cos(\gamma_H - \gamma_L)}{\cos\gamma_L \cos\gamma_H} \cdot \left((1-s)\tan\gamma_L \tan\frac{\pi - \gamma_H + \gamma_L}{2} - s \right).$$

This expression is positive if and only if the last factor is positive. The lemma now follows directly from the definition of γ_G in (147). ■

Remark: It follows from Lemma 39 that $f(s,\, \gamma_L,\, \gamma_L) > f(s,\, \gamma_L,\, \gamma_H)$ for all $\gamma_H \leqslant \pi/2$ whenever $s < \sin\gamma_L$.

TABLE VI
BOUNDS ON $A(n, 14, w)$

n	w = 8	9	10	11	12	13	14
16	2^{5}						
17	2^{5}						
18	2^{5}	2^{5}					
19	2^{5}	2^{10}					
20	2^{10}	2^{10}	2^{10}				
21	3^{5}	3^{5}	3^{5}				
22	3^{5}	3^{10}	4^{5}	4^{5}			
23	3^{5}	3^{10}	4^{10}	4^{10}			
24	3^{10}	4^{10}	5^{10}	6^{5}	6^{10}		
25	3^{10}	5^{5}	6^{10}	7^{10}	8^{10}		
26	4^{5}	6^{5}	8^{5}	10^{5}	13^{5}	14^{5}	
27	4^{10}	6^{10}	9^{5}	13^{10}	$19 - \mathbf{20}^{10}$	27^{5}	
28	4^{10}	7^{5}	11^{10}	21^{5}	$\mathbf{28}^{5}$	$\mathbf{28}^{5}$	54^{5}

The next lemma combines Lemmas 40–42 to summarize the bounds that hold for $\gamma_H < \pi/2$. There is an intentional overlap between some of the cases in the lemma.

Lemma 43: If $0 < \gamma_L \leqslant \gamma_H < \pi/2$, then

$$A_Z(n, s, \gamma_L, \gamma_H) \leqslant A_Z(n, s, \gamma_G, \gamma_H) + A_S(n-1, f(s, \gamma_L, \gamma_L)), \quad \text{if } \gamma_H > \gamma_G \tag{153}$$

$$A_Z(n, s, \gamma_L, \gamma_H) \leqslant A_S(n-1, f(s, \gamma_L, \gamma_H)), \quad \text{if } s < \cos(\gamma_H - \gamma_L) \text{ and } \gamma_H > \gamma_G \tag{154}$$

$$A_Z(n, s, \gamma_L, \gamma_H) \leqslant A_S(n-1, f(s, \gamma_L, \gamma_L)), \quad \text{if } s \geqslant -\cos 2\gamma_L \text{ and } \gamma_H \leqslant \gamma_G \tag{155}$$

$$A_Z(n, s, \gamma_L, \gamma_H) = 1, \qquad \text{if } s < -\cos 2\gamma_L. \tag{156}$$

Proof: The bounds (154) and (155) follow from Lemmas 41 and 42. Note that $\gamma_G < \gamma_H \leqslant \pi/2$ implies that $s > \sin\gamma_L$, in view of Lemma 39. This, in turn, is a stronger condition than $s \geqslant -\cos 2\gamma_L$. Hence, the constraint $s \geqslant -\cos 2\gamma_L$ in Lemma 41 would be redundant in (154). Similarly, the constraint $s < \cos(\gamma_H - \gamma_L)$ in Lemma 41 would be redundant in (155). This is so because if $s \geqslant \cos(\gamma_H - \gamma_L)$ and $\gamma_H \leqslant \gamma_G$, then $s > \sin\gamma_L$ and $s \geqslant \cos(\gamma_G - \gamma_L)$, which contradicts Lemma 39. The inequality (153) follows from

$$A_Z(n, s, \gamma_L, \gamma_H) \leqslant A_Z(n, s, \gamma_L, \gamma_G) + A_Z(n, s, \gamma_G, \gamma_H) \tag{157}$$

if $\gamma_H \geqslant \gamma_G$, where the first term can be bounded using (155). Finally, (156) follows directly from Lemma 40. ■

The next lemma gives upper bounds for $\gamma_H = \pi/2$.

Lemma 44: If $0 < \gamma_L \leqslant \pi/2$, then

$$A_Z(n, s, \gamma_L, \pi/2) \leqslant A_Z(n, s, \gamma_G, \pi/2) + A_S(n-1, f(s, \gamma_L, \gamma_L)), \quad \text{if } s \geqslant \sin\gamma_L \tag{158}$$

$$A_Z(n, s, \gamma_L, \pi/2) \leqslant A_S(n-1, f(s, \gamma_L, \gamma_L)), \quad \text{if } -\cos 2\gamma_L \leqslant s < \sin\gamma_L \tag{159}$$

$$A_Z(n, s, \gamma_L, \pi/2) = 1, \qquad \text{if } s < -\cos 2\gamma_L. \tag{160}$$

Proof: The bound (158) follows, if $s > \sin\gamma_L$, from (157) and (155). If $s = \sin\gamma_L$, then for any γ such that

$$\pi/4 + \gamma_L/2 < \gamma < \pi/2$$

we have

$$A_Z(n, s, \gamma_L, \pi/2) \leqslant A_Z(n, s, \gamma, \pi/2) + A_Z(n, s, \gamma_L, \gamma)$$

and (158) follows by applying (149) and (155), respectively, to the two terms. To prove (159) and (160), we observe that if $s < \sin\gamma_L$, then

$$\pi - \gamma_L - \arccos s < \pi/2 < \gamma_G.$$

Letting $\gamma_H = \pi/2$ in Lemma 40 and using (155) to bound $A_Z(n, s, \gamma_L, \pi - \gamma_L - \arccos s)$ completes the proof. ■

The last component in the proof of Theorem 31 is a lower bound, given in the next lemma. This lemma is the counterpart to Lemma 41: we now reverse the mapping in (152) to construct zonal codes from spherical codes.

Lemma 45: For all $-\pi/2 < \gamma_L \leqslant \gamma_H < \pi/2$ and all s in the range $s \geqslant -\cos 2\gamma_L$, we have

$$A_Z(n, s, \gamma_L, \gamma_H) \geqslant A_S(n-1, f(s, \gamma_L, \gamma_L)).$$

Proof: Let $\mathcal{C}_S$ be a spherical code with maximum cosine s' in an $(n-1)$-dimensional subspace of $\mathbb{R}^n$. Let $\boldsymbol{e} \in \mathbb{R}^n$ be a unit vector orthogonal to this subspace. For any given γ_L, we construct the code

$$\mathcal{C} \stackrel{\text{def}}{=} \{\boldsymbol{y}\cos\gamma_L + \boldsymbol{e}\sin\gamma_L : \boldsymbol{y} \in \mathcal{C}_S\}.$$

It is easy to verify that $\|\boldsymbol{x}\| = 1$ and $\boldsymbol{x}\cdot\boldsymbol{e} = \sin\gamma_L$ for all $\boldsymbol{x} \in \mathcal{C}$. Furthermore, $\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 \leqslant s$ for all distinct $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{C}$, where

$$s = s'\cos^2\gamma_L + \sin^2\gamma_L$$

or, equivalently, $s' = f(s, \gamma_L, \gamma_L)$. Hence $\mathcal{C}$ is a zonal code and $A_Z(n, s, \gamma_L, \gamma_H) \geqslant A_S(n-1, s')$. ■

Theorem 31 now follows by combining Lemmas 43–45.

ACKNOWLEDGMENT

The authors are indebted to T. Ericson, T. Etzion, K. Metsch, J. B. Shearer, N. J. A. Sloane, the reviewers, and, in particular, S. Litsyn, for helpful discussions and for pointing out relevant references. They also thank P. Boyvalenkov and P. Kazakov, who kindly sent their program for generating upper bounds on the size of spherical codes, and Zs. Nagy whose help in obtaining and translating Hungarian literature is much appreciated.

REFERENCES

[1] N. Q. A, L. Györfi and J. L. Massey, "Constructions of binary constant-weight cyclic codes and cyclically permutable codes," *IEEE Trans. Inform. Theory*, vol. 38, pp. 940–949, May 1992.

[2] J. Aczél and T. Szele, "Solution 35" (in Hungarian), *Matematikai Lapok*, vol. 3, no. 1, pp. 94–95, 1952.

[3] E. Agrell, A. Vardy, and K. Zeger. Tables of binary block codes. [Online]. Available: http;//www.chl.chalmers.se/~agrell

[4] A. Ashikhmin, A. Barg, and S. Litsyn, "New upper bounds on generalized weights," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1258–1263, May 1999.

[5] A. E. Ashikhmin, I. S. Honkala, T. K. Laihonen, and S. N. Litsyn, "On relations between covering radius and dual distance," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1808–1816, Sept. 1999.

[6] L. A. Bassalygo, "New upper bounds for error correcting codes," *Probl. Inform. Transm.*, vol. 1, no. 4, pp. 32–35, 1968. (Original appearance in Russian in *Probl. Pered. Inform.*, vol. 1, pp. 41–44, Oct.–Dec. 1965).

[7] E. R. Berger, "Some additional upper bounds for fixed-weight codes of specified minimum distance," *IEEE Trans. Inform. Theory*, vol. IT-13, pp. 307–308, Apr. 1967.

[8] M. R. Best, A. E. Brouwer, F. J. MacWilliams, A. M. Odlyzko, and N. J. A. Sloane, "Bounds for binary codes of length less than 25," *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 81–93, Jan. 1978.

[9] S. Bitan and T. Etzion, “Constructions for optimal constant-weight cyclically permutable codes and difference families,” *IEEE Trans. Inform. Theory*, vol. 41, pp. 77–87, Jan. 1995.

[10] R. E. Blahut, *Theory and Practice of Error Control Codes*. Reading, MA: Addison-Wesley, 1983.

[11] S. Blake-Wilson and K. T. Phelps, “Constant weight codes and group divisible designs,” *Des., Codes Cryptogr.*, vol. 16, pp. 11–27, Jan. 1999.

[12] M. Blaum and J. Bruck, “Coding for tolerance and detection of skew in parallel asynchronous communications,” *IEEE Trans. Inform. Theory*, vol. 46, pp. 0000–0000, Nov. 2000.

[13] A. Blokhuis, A. R. Calderbank, K. Metsch, and A. Seress, “An embedding theorem for partial linear spaces,” in *Advances in Finite Geometries and Designs. Proc. Third Isle of Thorns Conf. 1990*, J. W. P. Hirschfeld, D. R. Hughes, and J. A. Thas, Eds. Oxford, U.K.: Oxford Univ. Press, 1991, pp. 35–46.

[14] R. C. Bose and W. S. Connor, “Combinatorial properties of group divisible incomplete block designs,” *Ann. Math. Statist.*, vol. 23, no. 3, pp. 367–383, 1952.

[15] P. G. Boyvalenkov, D. P. Danev, and S. P. Bumova, “Upper bounds on the minimum distance of spherical codes,” *IEEE Trans. Inform. Theory*, vol. 42, pp. 1576–1581, Sept. 1996.

[16] A. E. Brouwer, “$A(17, 6, 4) = 20$ or the nonexistence of the scarce design $SD(4, 1; 17, 21)$,” Stichting Mathematisch Centrum, Amsterdam, The Netherlands, Rep. ZW 62/75, Dec. 1975.

[17] A. E. Brouwer, J. B. Shearer, N. J. A. Sloane, and W. D. Smith, “A new table of constant weight codes,” *IEEE Trans. Inform. Theory*, vol. 36, pp. 1334–1380, Nov. 1990.

[18] F. Buekenhout, Ed., *Handbook of Incidence Geometry*. Amsterdam, The Netherlands: Elsevier, 1995.

[19] F. R. K. Chung, J. A. Salehi, and V. K. Wei, “Optical orthogonal codes: Design, analysis, and applications,” *IEEE Trans. Inform. Theory*, vol. 35, pp. 595–604, May 1989.

[20] M. J. Colbourn, “Some new upper bounds for constant weight codes,” *IEEE Trans. Inform. Theory*, vol. IT-26, p. 478, July 1980.

[21] W. S. Connor Jr., “On the structure of balanced incomplete block designs,” *Ann. Math. Statist.*, vol. 23, pp. 57–71, Mar. 1952.

[22] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed. New York: Springer, 1999.

[23] H. Davenport and G. Hajós, “Problem 35” (in Hungarian), *Matematikai Lapok*, vol. 2, no. 1, p. 68, 1951.

[24] Ph. Delsarte, “An algebraic approach to the association schemes of coding theory,” *Philips Res. Repts. Suppl.*, no. 10, 1973.

[25] J. Dinitz and C. J. Colburn, *Handbook on Combinatorial Designs*. Boca Raton, FL: CRC, 1996.

[26] P. Erdös, “Problem 20” (in Hungarian), *Matematikai Lapok*, vol. 1, no. 3, p. 226, 1950.

[27] T. Ericson and V. Zinoviev, “Spherical codes generated by binary partitions of symmetric pointsets,” *IEEE Trans. Inform. Theory*, vol. 41, pp. 107–129, Jan. 1995.

[28] ——, *Codes on Euclidean Spheres*. Amsterdam, The Netherlands: Elsevier, 2000.

[29] T. Etzion, “Constructions of error-correcting DC-free block codes,” *IEEE Trans. Inform. Theory*, vol. 36, pp. 889–905, July 1990.

[30] C. V. Freiman, “Upper bounds for fixed-weight codes of specified minimum distance,” *IEEE Trans. Inform. Theory*, vol. IT-10, pp. 246–248, July 1964.

[31] R. L. Graham and N. J. A. Sloane, “Lower bounds for constant weight codes,” *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 37–43, Jan. 1980.

[32] M. Hall Jr., *Combinatorial Theory*, 2nd ed. New York: Wiley, 1986.

[33] R. W. Hamming, “Error detecting and error correcting codes,” *Bell Syst. Tech. J.*, vol. 26, pp. 147–160, Apr. 1950.

[34] I. Honkala, “Bounds for binary constant weight and covering codes,” Licentiate thesis, Dept. Math., Univ. of Turku, Turku, Finland, Mar. 1987.

[35] S. M. Johnson, “A new upper bound for error-correcting codes,” *IRE Trans. Inform. Theory*, vol. IT-8, pp. 203–207, Apr. 1962.

[36] ——, “On upper bounds for unrestricted binary error-correcting codes,” *IEEE Trans. Inform. Theory*, vol. IT-17, pp. 466–478, July 1971.

[37] ——, “Upper bounds for constant weight error correcting codes,” *Discr. Math.*, vol. 3, pp. 109–124, 1972.

[38] C. W. H. Lam, L. Thiel, and S. Swiercz, “The non-existence of finite projective planes of order 10,” *Canad. J. Math.*, vol. 41, pp. 1117–1123, 1989.

[39] V. I. Levenshtein, “Upper-bound estimates for fixed-weight codes,” *Probl. Inform. Transm.*, vol. 7, no. 4, pp. 281–287, Jan. 1974. (Original appearance in Russian in *Probl. Pered. Inform.*, vol. 7, pp. 3–12, Oct.–Dec. 1971).

[40] ——, “On bounds for packings in n-dimensional Euclidean space,” *Sov. Math.–Dokl.*, vol. 20, no. 2, pp. 417–421, Mar.–Apr. 1979. (Original appearance in Russian in *Dokl. Akad. Nauk SSSR*, vol. 245, pp. 1299–1303, Mar.–Apr. 1979).

[41] ——, “Universal bounds for codes and designs,” in *Handbook of Coding Theory*, V. S. Pless and W. C. Huffman, Eds. Amsterdam, The Netherlands: Elsevier, 1998, vol. 1, pp. 499–648.

[42] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. Amsterdam, The Netherlands: North-Holland, 1977.

[43] R. J. McEliece, E. R. Rodemich, H. Rumsey Jr., and L. R. Welch, “New upper bounds on the rate of a code via the Delsarte–MacWilliams inequalities,” *IEEE Trans. Inform. Theory*, vol. IT-23, pp. 157–166, Mar. 1977.

[44] M. Plotkin, “Binary codes with specified minimum distance,” *IRE Trans. Inform. Theory*, vol. IT-6, pp. 445–450, Sept. 1960.

[45] C. L. M. van Pul, “On bounds on codes,” Master’s thesis, Dept. Math. Comput. Sci., Eindhoven Univ. Technol., Eindhoven, The Netherlands, Aug. 1982.

[46] L. Råde and B. Westergren, *Mathematics Handbook for Science and Engineering*. Basel, Switzerland: Birkhäuser, 1995.

[47] R. A. Rankin, “The closest packing of spherical caps in n dimensions,” *Proc. Glasgow Math. Assoc.*, vol. 2, pp. 139–144, 1954–1956.

[48] A. Samorodnitsky, “On the optimum of Delsarte’s linear program,” *J. Comb. Theory*, ser. A, to be published.

[49] J. B. Shearer, private communication, Feb. 2000.

[50] K. Sarkadi and T. Szele, “Solution 20” (in Hungarian), *Matematikai Lapok*, vol. 2, no. 1, pp. 76–77, 1951.

[51] D. R. Stinson, “Determination of a packing number,” *Ars Combin.*, vol. 3, pp. 89–114, 1977.

[52] H. C. A. van Tilborg and M. Blaum, “On error-correcting balanced codes,” *IEEE Trans. Inform. Theory*, vol. 35, pp. 1091–1095, Sept. 1989.

[53] V. D. Tonchev, *Combinatorial Configurations. Designs, Codes, Graphs*. Harlow, U.K.: Longman Scientific & Technical, 1988.

[54] X.-M. Wang and Y.-X. Yang, “On the undetected error probability of nonlinear binary constant weight codes,” *IEEE Trans. Commun.*, vol. 42, pp. 2390–2394, July 1994.

[55] N. Wax, “On upper bounds for error detecting and error correcting codes of finite length,” *IRE Trans. Inform. Theory*, vol. IT-5, pp. 168–174, Dec. 1959.

[56] D. Zwillinger, Ed., *CRC Standard Mathematical Tables and Formulae*, 30th ed. Boca Raton, FL: CRC, 1996.