Asymptotic Capacity of Two-dimensional Channels with Checkerboard Constraints[1]

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One-dimensional channels satisfying run length constraints are important in magnetic recording applications and two-dimensional channels satisfying run length constraints have been considered in relation to optical recording applications (see the references in [1]). One-dimensional (d,k) run length constraints require that in any binary sequence, there be at least d and at most k 0s between consecutive 1s. Two-dimensional run length constraints require that one-dimensional run length constraints be satisfied both horizontally and vertically in a two-dimensional rectangular binary array.

In addition to run length constraints, other types of constraints can be used to model certain two-dimensional channels. An example of a circularly symmetric two-dimensional constraint occurs by requiring that any point in the two-dimensional $\mathbf{Z}^2$ lattice be labeled 0 if it is within a prescribed distance from a lattice point with label 1.

One could alternatively require that every 1 be surrounded by 0s falling in a given sized hexagon, square, or more generally any other shape of interest. In general, a large class of such two-dimensional constraints can be characterized by some bounded measurable two-dimensional set S, and the requirement that for every 1 stored in the plane, it must at least be surrounded by a set of 0s arranged in the shape of S. Such a code is said to satisfy the constraint S. These constraints are known as checkerboard constraints [3].

For a convex symmetric checkerboard constraints S, we determine the rate at which the capacity goes to zero, as a function of the area $A(S)$ of the constraint. It is shown that as $A(S) \to \infty$, the capacity decays to zero at the rate $4\delta(S)(\log_2 A(S))/A(S)$, where $\delta(S)$ is the packing density of the set S. Thus, for example, since the packing density (in the plane) of squares or hexagons is $\delta(S) = 1$, this implies that the capacity of two-dimensional channels satisfying square or hexagon checkerboard constraints is asymptotically equal to $4(\log_2 A(S))/A(S)$ as the area grows without bound. Similarly, if S is a circular constraint, then the asymptotic capacity is $\frac{2\pi}{\sqrt{3}}(\log_2 A(S))/A(S)$ since $\delta(S) = \pi/(2\sqrt{3})$.

I. Preliminaries

Given a set $S \subset \mathbf{R}^2$, any function $f : S \cap \mathbf{Z}^2 \to \{0,1\}$ is called a *labeling* of S. For any set $S \subset \mathbf{R}^2$, let $A(S)$ be the area of S and let $\Lambda(S) = \left|S \cap \mathbf{Z}^2\right|$ be the number of $\mathbf{Z}^2$-lattice points contained in S. A set $S \subset \mathbf{R}^2$ is *symmetric* if $x \in S \Leftrightarrow -x \in S$.

Given a set $V \subset \mathbf{R}^2$ and a checkerboard constraint S, a labeling f of V is said to be S-*valid* on V if $f(y) = 0$ whenever $f(x) = 1$, for all $x \in V \cap \mathbf{Z}^2$ and $y \in (x+S) \cap (V \setminus \{x\}) \cap \mathbf{Z}^2$. That is, f satisfies the checkerboard constraint S on the set $V \subset \mathbf{R}^2$. The number of S-valid labelings of a set $V \subset \mathbf{R}^2$ is denoted by $N_S(V)$. The *capacity* C_S corresponding to the checkerboard constraint S is

$$C_S = \lim_{\kappa,\lambda,\mu,\nu\to\infty} \frac{\log_2 N_S\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}{A\left(R^{(\mu,\nu)}_{(-\kappa,-\lambda)}\right)}. \tag{1}$$

A proof given in [2] shows that the above limit exists.

The capacities of various channels satisfying convex checkerboard constraints were studied in [3]. These included the diamond, hexagonal, square, and (d,∞) run length checkerboard constraints.

Every checkerboard constraint S is equivalent to the symmetric checkerboard constraint $S \cup -S$ in the sense that the sets of S-valid labelings and $(S \cup -S)$-valid labelings of any set $V \subset \mathbf{R}^2$ are identical. Thus no generality is lost if we restrict attention to symmetric checkerboard constraints when computing capacities.

Notation: Let U be the set of all checkerboard constraints and let $f : U \to \mathbf{R}$. For any $S \in U$ and $L \in \mathbf{R}$, we write $\lim_{A(S)\to\infty} f(S) = L$ to mean that $\lim_{\alpha\to\infty} f(\alpha S) = L$. That is, the set S is inflated without bound by the factor α but retains the same shape.

Theorem 1 *If S is an open convex symmetric checkerboard constraint with area $A(S)$, capacity C_S, and packing density $\delta(S)$, then*

$$\lim_{A(S)\to\infty} C_S \cdot \frac{A(S)}{\log_2 A(S)} = 4\delta(S). \tag{2}$$

In fact, a more general rate of convergence can be obtained for the capacity of two-dimensional channels with checkerboard constraints whose interior contains the origin, but without exactly identifying the convergence constant. Such constraints are not necessarily convex. The capacity is shown in Theorem 2 below to still decay asymptotically at the rate $(\log A(S))/A(S)$ in these cases. Theorem 2 makes precise a prediction given in [3]: "Intuitively, we expect that the capacity of a given constraint will be inversely proportional to the number of *zeros* in the constraint."

Theorem 2 *If S is a checkerboard constraint whose interior contains the origin, then*

$$0 < \liminf_{A(S)\to\infty} C_S \cdot \frac{A(S)}{\log_2 A(S)} \le \limsup_{A(S)\to\infty} C_S \cdot \frac{A(S)}{\log_2 A(S)} < \infty.$$

References

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