Performance of Quantizers on Noisy Channels using Structured Families of Codes *

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Abstract

Achievable distortion bounds are derived for the cascade of structured families of binary linear channel codes and binary lattice vector quantizers. It is known that for the cascade of asymptotically good channel codes and asymptotically good vector quantizers the end-to-end distortion decays to zero exponentially fast as a function of the overall transmission rate, and is achieved by choosing a channel code rate that is independent of the overall transmission rate. We show that for certain families of practical channel codes and binary lattice vector quantizers, the overall distortion can still be made to decay to zero exponentially fast as the transmission rate grows, although the exponent is a sub-linear function of the transmission rate. This is achieved by carefully choosing a channel code rate that decays to zero as the transmission rate grows. Explicit channel code rate schedules are obtained for several well-known families of channel codes.

1 Introduction

We exploit results from high resolution theory to obtain new quantization results for noisy channels. High resolution quantization theory for noisy channels gives analytic descriptions of the minimum achievable average distortion, as a function of the transmission rate, the source density, and the vector dimension. For distortion functions which are powers of Euclidean distances and with no channel noise, the minimum average distortion is known to decay to zero exponentially fast as the transmission rate increases [1]. It was shown in [2, 3] that when the source information is transmitted across a noisy channel, the minimum average distortion again decays to zero exponentially fast as the transmission rate increases, although the exponential decay constant is reduced by an amount dependent on how poor the channel is. In fact, the rate of decay of distortion in the noisy channel case is closely related to the optimal allocation of transmission rate between source coding and channel coding (via the channel code rate).

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The results in [3] provide mathematical guarantees for a potentially achievable minimum quantizer distortion in the presence of channel noise. However, those results assume the existence of optimal channel codes, namely those described in Shannon's channel coding theorem using random coding arguments. Similar techniques were used to generalize the results of [3] to Gaussian channels [4] and to certain algebraic geometry codes [5]. Hence, the results in [3–5] are existence constructions and do not necessarily correspond to achievable performance based on the best presently known implementable channel codes. There is thus motivation to find a high resolution theory for quantization with a noisy channel, using families of structured algebraic channel codes.

However, finding such a high resolution theory appears to be a difficult task for general unstructured source coders, even if the channel coders are structured. In this paper, we approach the problem by examining systems with structure in both the source coder and channel coder. Such systems are practical to implement and also give insight (via distortion bounds) into the unstructured source coder case.

To illustrate the problem at hand by way of an example, suppose a random variable uniformly distributed on $[0, 1]$ is uniform scalar quantized, and transmitted across a binary symmetric channel using a repetition code. For a fixed number of available bits R per transmission, how many times should each information bit be repeated in the repetition code to minimize the end-to-end mean squared error? In other words, what is the optimal rate allocation between source and channel coding? If the channel code rate is decreased, fewer uncorrected bit errors occur but at the expense of coarser quantization, and vice versa if the channel code rate is increased.

A key assumption in [3, 5] is that by keeping the channel code rate fixed (below capacity) while increasing the overall transmission rate R, the probability of decoding error P_e can decay to zero exponentially fast as a function of R. This assumption is valid for "Shannon-optimal" codes and more generally for asymptotically good codes, but most known structured families of channel codes (e.g. Hamming, BCH, Reed-Muller) do not have this property. In the repetition code example, keeping the channel code rate fixed is equivalent to keeping the number of repetitions constant. This in turn implies that the probability of incorrectly decoding an information bit does not change. Therefore, P_e is bounded away from zero, since the probability of decoding error (i.e. an incorrect block) is at least as large as the probability of a single bit error. In this paper, we investigate the rate allocation problem for structured families of source coders which are asymptotically good and for structured families of channel coders which are not asymptotically good, but which can be used in practice.

A common method for lossy transmission of source data across a noisy channel uses independently designed source coders and channel coders. This follows Shannon's basic "separation principle" in source and channel coding, which is known to be optimal for asymptotically large blocklengths. An important design parameter is the allocation of the available transmission rate between source and channel coding. Tight upper and lower bounds on the optimal tradeoff between source and channel coding are known for certain codes and channels and pth-power distortion measures [2–5]. These results exploit the fact that the distortion contributions of optimal source coding and optimal channel coding decay exponentially fast as functions of the overall transmission rate. The source coder is taken to be a 'good' vector quantizer (one that obeys Zador's decay rate) in [2–5], and index assignment randomization is used. In both [3] and [4], the channel codes are assumed to have exponentially decaying error probabilities achieving the expurgated error exponent for the given channel (a binary symmetric channel in [3] and an additive white Gaussian noise channel in [4]). Although such codes are known to exist, no efficiently decodable ones have yet been discovered. In [5], the results of [3] are extended to q-ary symmetric channels, and a class of asymptotically good channel codes (namely those attaining the Gilbert-Varshamov and Tsfasman-Vlădut-Zink bounds) is examined. Constructions of channel codes better than the Gilbert-Varshamov bound are known [6, 7], but the best known algorithms are not currently practical.

The channel codes considered in [3–5] all have the property that their channel code rates are bounded away from zero for increasing blocklengths. In the present paper we investigate the tradeoff between source and channel coding for structured classes of codes whose channel code rates approach zero in the limit as the blocklength grows. Hence, we seek a decay schedule of the channel code rate as a function of the overall transmission rate which minimizes the overall distortion. The channel codes we examine are classical binary linear block codes including repetition codes, Reed-Muller codes, and BCH codes. We call (as in [8]) the structured source coders in this paper Binary Lattice Vector Quantizers. Vector quantizers with essentially identical structure have been extensively studied under various different names in [8–11].

The main results of this paper are collected into Theorem 1 in Section 3, which gives achievable bounds on the asymptotic mean squared error performance of binary lattice vector quantizers and several useful families of binary linear block channel codes on a binary symmetric channel. The bounds in Theorem 1 show that the minimum distortion with certain structured codes decays to zero as $O\left(2^{-2Rg(R)}\right)$, where $g(R) \to 0$ as $R \to \infty$. The distortion bounds are obtained by choosing $g(R) = O\left(\frac{1}{\sqrt{R}}\right)$ for repetition codes and $g(R) = O\left(\sqrt{\frac{(\log R)^l}{R}}\right)$ for Reed-Muller codes and duals of BCH codes. The constants inside the $O(\cdot)$ depend on the channel noise level. In contrast, for optimal unstructured vector quantizers and no channel noise, $g(R) = 1$ for all R, and for optimal unstructured vector quantizers and optimal channel codes on a noisy channel, $g(R) < 1$ (depending on the channel noise level) and g is bounded away from zero. Since structured source coders are assumed in this paper, the distortion bounds given are also upper bounds on the distortion using optimal unstructured VQ with the same structured channel codes. Section 2 gives the framework for the source/channel coding problem and Section 3 gives the results of the paper.

2 The Cascaded System

Definition 1 A d-dimensional, 2^k-point *noisy channel vector quantizer* with *index set* $\mathbb{Z}_2^k$, *codebook* $\mathcal{Y}$, and with an $[n, k]$ linear channel code $\mathcal{C}$ operating on a binary channel, is a functional composition $\mathcal{Q} = \mathcal{D}_Q \circ \mathcal{D}_C \circ \eta \circ \mathcal{E}_C \circ \mathcal{E}_Q$, where $\mathcal{E}_Q \colon \mathbb{R}^d \to \mathbb{Z}_2^k$

is a quantizer encoder, $\mathcal{D}_Q\colon \mathbb{Z}_2^k \to \mathcal{Y}$ is a quantizer decoder, $\mathcal{E}_C\colon \mathbb{Z}_2^k \to \mathcal{C}$ is a channel encoder, $\mathcal{D}_C\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^k$ is a channel decoder, and $\eta\colon \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ is a random mapping representing a noisy channel. The overall *transmission rate* of a noisy channel vector quantizer is given by $R = n/d$. The *source coding rate* (or resolution) of the noiseless vector quantizer $\mathcal{Q}_0 = \mathcal{D}_Q \circ \mathcal{E}_Q$ is defined as $R_S = k/d$.

The mean squared distortion of a noisy channel vector quantizer for a source random variable $\mathbf{X} \in \mathbb{R}^d$ is

$$\Delta = \mathrm{E}\, \|\mathbf{X} - \mathcal{Q}(\mathbf{X})\|^2 . \tag{1}$$

We define the *source distortion* (the distortion on a noiseless channel, due to quantization only) as

$$\Delta_S = \mathrm{E}\, \|\mathbf{X} - \mathcal{Q}_0(\mathbf{X})\|^2 , \tag{2}$$

and the *channel distortion* (the component of the distortion influenced by channel errors) as

$$\Delta_C \stackrel{\triangle}{=} \mathrm{E}\, \|\mathcal{Q}_0(\mathbf{X}) - \mathcal{Q}(\mathbf{X})\|^2 . \tag{3}$$

The Minkowski inequality can be used to bound the distortion as

$$\Delta \leq \left(\sqrt{\Delta_S} + \sqrt{\Delta_C}\right)^2 . \tag{4}$$

The high resolution (i.e. large R_S) behavior of Δ_S for optimal quantization of a bounded source is described by Zador's formula, which is stated below in a convenient form.

Lemma 1 (Zador [1]) *The minimum mean squared error of a rate R_S vector quantizer is asymptotically (as $R_S \to \infty$) given by*

$$\Delta_S = 2^{-2R_S + O(1)}. \tag{5}$$

Lemma 1 is often referred to as the "6 dB per bit" rule, since

$$10 \log_{10} \left(2^{-2R_S + O(1)} / 2^{-2(R_S+1)+O(1)}\right) \to 20 \log_{10} 2 \approx 6 \text{ dB}.$$

We say that a sequence of quantizers is *asymptotically good* if

$$\limsup_{R_S \to \infty} \Delta_S 2^{2R_S} < \infty. \tag{6}$$

Lemma 1 shows that optimal quantizers are asymptotically good. In fact, a large class of quantizers including uniform quantizers and other lattice-based vector quantizers are also asymptotically good, although the limit in (6) may be larger than for optimal quantizers.

The asymptotic behavior of Δ_C with increasing R is affected by the error-correcting capabilities of the channel codes used. The channel distortion using optimal quantization is bounded away from zero for non-redundant channel codes [2] and decays to zero exponentially fast for "Shannon-optimal" channel codes [3] and asymptotically good channel codes [5]. To show that Δ_C can also be made to decay to zero exponentially fast as a function of R for classical linear block channel codes, the vector quantizers in this paper are chosen to be bounded binary lattice vector quantizers. This allows a simple bound on Δ_C.

We call a sequence of quantizers *bounded*, if the codepoints of the quantizers are bounded, that is,

$$\sup_k \left(\max_{\mathbf{y} \in \mathcal{Y}_k} \|\mathbf{y}\| \right) < \infty, \tag{7}$$

where $\mathcal{Y}_k$ denotes the codebook of the k-bit quantizer in the sequence. Unrestricted optimal quantizers for a bounded source are bounded, as are large classes of other useful quantizers including truncated lattice VQs for example.

2.1 Binary Lattice VQ

Definition 2 For positive integers d and k, a d-dimensional, 2^k-point *binary lattice vector quantizer* is a vector quantizer with index set $\mathbb{Z}_2^k$, whose codebook contains codevectors of the form

$$\mathbf{y}_i = \mathbf{y}_0 + \sum_{l=0}^{k-1} \mathbf{v}_l i_l \qquad \forall i \in \mathbb{Z}_2^k, \tag{8}$$

where $\mathbf{y}_0 \in \mathbb{R}^d$ is an *offset vector*, $\{\mathbf{v}_l\}_{l=0}^{k-1} \subset \mathbb{R}^d$ is the set of *generator vectors*, ordered by $\|\mathbf{v}_0\| \leq \|\mathbf{v}_1\| \leq \ldots \leq \|\mathbf{v}_{k-1}\|$, and i_l denotes the l^{th} bit of the index i.

In this paper, we focus on Binary Lattice Vector Quantizers (BLVQ). There are several equivalent formulations of BLVQ as, for example, truncated lattice VQ, direct sum (or residual) VQ, and VQ by a Linear Mapping of a (non-redundant) Block Code. BLVQs can save in memory requirements and encoding complexity. They can also be used for progressive transmission and possess a certain natural robustness to channel noise (see [8] for details).

BLVQs encompass a broad class of useful structured quantizers. For example, a 2^k-level uniform scalar quantizer on the interval (a, b) is a special case of a binary lattice quantizer, obtained by setting $y_0 = a + s/2$ and $v_l = 2^l s$, where $s = (b-a)2^{-k}$ denotes the quantizer stepsize. As a consequence, sequences of asymptotically good BLVQs exist. In fact, for any bounded source, a sequence of increasingly finer (properly truncated and rotated) cubic lattices containing the support of the source is both bounded and asymptotically good. Thus, in what follows, we restrict attention to asymptotically good bounded sequences of binary lattice vector quantizers.

2.2 Linear Codes on a Binary Symmetric Channel

Definition 3 A *linear binary* $[n, k, d_{\min}]$ *block channel code* is a linear subspace of $\mathbb{Z}_2^n$ containing 2^k binary n-tuples called *codewords*, each with at least $d_{\min}$ nonzero components. The *channel code rate* is given by $r = k/n$, and the *relative minimum distance* by $\delta = d_{\min}/n$.

To obtain asymptotic results we consider families of $[n, k, d_{\min}]$ linear channel codes indexed by the blocklength n. All families of channel codes fall into exactly one of the following three categories (assuming the limits of $d_{\min}/n$ and k/n exist as $n \to \infty$):

- $\lim_{n\to\infty} \frac{d_{\min}}{n} = 0$
  The best known families of block channel codes in this category have $k/n \to 1$ as $n \to \infty$. Examples include Hamming codes, families of t-error-correcting binary BCH codes for any fixed t, and lth-order Reed-Muller codes if l is an increasing function of the blocklength. From a source-channel tradeoff perspective, the best codes in these families are those with small blocklengths. Hence, these codes are not relevant to our asymptotic investigations, although their duals are.

- $\lim_{n\to\infty} \frac{d_{\min}}{n} > 0$ and $\lim_{n\to\infty} \frac{k}{n} > 0$
  Families of codes with both their rate and relative minimum distance bounded away from 0 are called *asymptotically good* [12]. Examples include Justesen codes [12, p. 306 ff] and codes satisfying the Zyablov bound [12, p. 315], the Gilbert-Varshamov bound [12, p. 557], or the Tsfasman-Vlădut-Zink bound [13]. Bounds on the asymptotically optimal source/channel rate allocation were derived in [5] for some of these codes.

- $\lim_{n\to\infty} \frac{d_{\min}}{n} > 0$ and $\lim_{n\to\infty} \frac{k}{n} = 0$
  Codes that fall into this category include repetition codes, lth-order Reed-Muller codes for any fixed order l, t-error-correcting binary BCH codes with $t = O(n)$, and duals of t-error-correcting binary BCH families for any fixed t. The probability of decoding error decays to zero exponentially fast for families of this type. Since $k/n \to 0$, relatively less information is transmitted as the blocklength increases, but more reliably.

In this paper, we focus attention on the third category above. One seeks an optimal "schedule" of the rate k/n converging to 0 as a function of the blocklength n.

2.3 Rate Allocation for BLVQ

Since the codevectors of a BLVQ are linear combinations of their respective index bits, it follows that the channel distortion of a bounded sequence of BLVQs is uniformly bounded (in R) as

$$\Delta_C \leq c P_{\max}^{(\mathrm{bit})}, \tag{9}$$

where c is a finite constant (independent of R) and $P_{\max}^{(\text{bit})}$ is the largest of the error probabilities for a received (channel decoded) index bit. Thus, we need to show that $P_{\max}^{(\text{bit})}$ can be made to go to zero exponentially fast as a function of the overall transmission rate R.

We consider a family of $[n, k, d_{\min}]$ channel codes satisfying $\lim_{n\to\infty} k/n = 0$ and $\lim_{n\to\infty} d_{\min}/n > 2\epsilon > 0$, where ϵ is the crossover probability of the underlying binary symmetric channel. We further assume that k is a monotone increasing function of n, which implies a one-to-one relationship between the channel code rate r and the blocklength n (e.g. this holds for repetition codes and Hamming codes). We divide the Rd bits per sample into blocks of shorter channel codes from the same family of $[n, k, d_{\min}]$ codes, and assume that each has the same blocklength n (a divisor of Rd). Thus, the length Rd channel code is the (Rd/n)-ary Cartesian product of identical length n codes. This maintains the overall transmission rate R bits per vector component, and allows a variety of channel code rates r.

The total distortion can be shown to be bounded as

$$\Delta \le \left(2^{-R\frac{k}{n}+O(1)} + 2^{-\frac{n}{2}\mathcal{D}\left(\frac{d_{\min}}{2n}\middle\|\epsilon\right)+O(1)}\right)^2, \tag{10}$$

where $\mathcal{D}\left(\delta\|\,\epsilon\right) = \delta\log_2\frac{\delta}{\epsilon} + (1-\delta)\log_2\frac{1-\delta}{1-\epsilon}$ is the binary relative entropy function (information divergence).

The value of the right side of (10) for any n that divides Rd represents an achievable distortion, since there exist binary lattice vector quantizers and families of channel codes that satisfy such a bound. In particular, we minimize the right side of (10) over n. Let n_R denote a value of n which achieves the minimum. Asymptotically (in R), $n_R \to \infty$ must hold, for otherwise the second term in (10) would be bounded away from zero. In fact, to minimize the bound in (10) the exponents of the two decaying exponentials have to be asymptotically equal. Since $n_R \to \infty$ as $R \to \infty$ and the families of codes considered satisfy $\lim_{n\to\infty} d_{\min}/n > 2\epsilon$ by assumption, the limit of the information divergence in the exponent of the second term in (10) is a finite non-zero constant which we denote by $\beta \triangleq \mathcal{D}\left(\frac{1}{2}\lim_{n\to\infty}\frac{d_{\min}}{n}\middle\|\,\epsilon\right)$. Thus, the asymptotically minimizing n_R satisfies

$$\lim_{R\to\infty}\frac{2Rk}{n_R^2\beta} = 1. \tag{11}$$

Let r_R denote the channel code rate corresponding to the n_R which solves (11). Then by (10), the overall distortion vanishes at least as fast as $2^{-2Rr_R+O(1)}$. The next section presents the rate allocations r_R obtained from solutions to (11) for various code families.

3 Asymptotic Distortion Decay Rates

Theorem 1 *Let $\mathbf{X} \in \mathbb{R}^d$ be a bounded random variable which is transmitted at a rate R bits per component across a binary symmetric channel with crossover probability ϵ.*

Suppose the source coder is chosen from a sequence of asymptotically good bounded binary lattice vector quantizers, and the channel coder is chosen from a family of $[n, k, d_{\min}]$ *linear block channel codes satisfying* $\lim_{n\to\infty} k/n = 0$ *and* $\lim_{n\to\infty} d_{\min}/n > 2\epsilon$*. Then, the overall minimum mean squared error decays (asymptotically in R) at least as fast as*

$$\Delta \leq 2^{-2Rr_R+O(1)}, \tag{12}$$

which is achieved by a channel code rate r_R*, for various channel code families as follows:*

(*i*) *for a family of* $[n, 1, n]$ *repetition codes* $(n \geq 1)$

$$r_R = \sqrt{\frac{-\log_2 2\sqrt{\epsilon(1-\epsilon)}}{2R}}, \qquad \epsilon \in (0, 1/2); \tag{13}$$

(*ii*) *for a family of lth-order* $\left[2^m, \sum_{i=0}^{l}\binom{m}{i}, 2^{m-l}\right]$ *Reed-Muller codes* $(m \geq 1)$

$$r_R = \sqrt{\frac{-\left(\log_2 2^{l+1}\left(\epsilon\left(\frac{1-\epsilon}{2^{l+1}-1}\right)^{2^{l+1}-1}\right)^{\frac{1}{2^{l+1}}}\right)(\log_2 R)^l}{l!\ 2^{l+1}R}}, \qquad \epsilon \in (0, 1/2^{l+1}); \tag{14}$$

(*iii*) *and for a family of duals of extremal t-error-correcting* $[2^m - 1, mt, 2^{m-1} - \lfloor\log_2(2t-1)\rfloor]$ *BCH codes* $(m \geq 1)$

$$r_R = \sqrt{\frac{-t\left(\log_2 4\left(\epsilon\left(\frac{1-\epsilon}{3}\right)^3\right)^{\frac{1}{4}}\right)\log_2 R}{4R}}, \qquad \epsilon \in (0, 1/4). \tag{15}$$

Figure 1 provides an illustration of Theorem 1 for the special case of using a uniform scalar quantizer for a uniform source on $(0, 1)$ and a family of repetition codes on a binary symmetric channel with $\epsilon = 10^{-3}$. For each $R = 1, 2, 3, \ldots, 128$, the repetition code with the smallest distortion was found by exhaustive search and the resulting rate was plotted (discrete dots). Since deleting a bit of an even length repetition code results in an odd length repetition code with the same bit error probability, using the extra bit for source coding always results in a smaller overall distortion. Hence, in addition to the analytic expression for r_R from (13) (dashed curve), we also plotted the channel code rate corresponding to the closest odd blocklength (step function).

As with Zador's lemma, Theorem 1 also gives a rule of thumb for the expected gain in system performance per bit increase in the overall transmission rate. Unlike on an error-free channel or on a noisy channel using asymptotically good codes (as in [3–5]), however, there is no fixed increase in the signal-to-noise ratio per "bit investment".

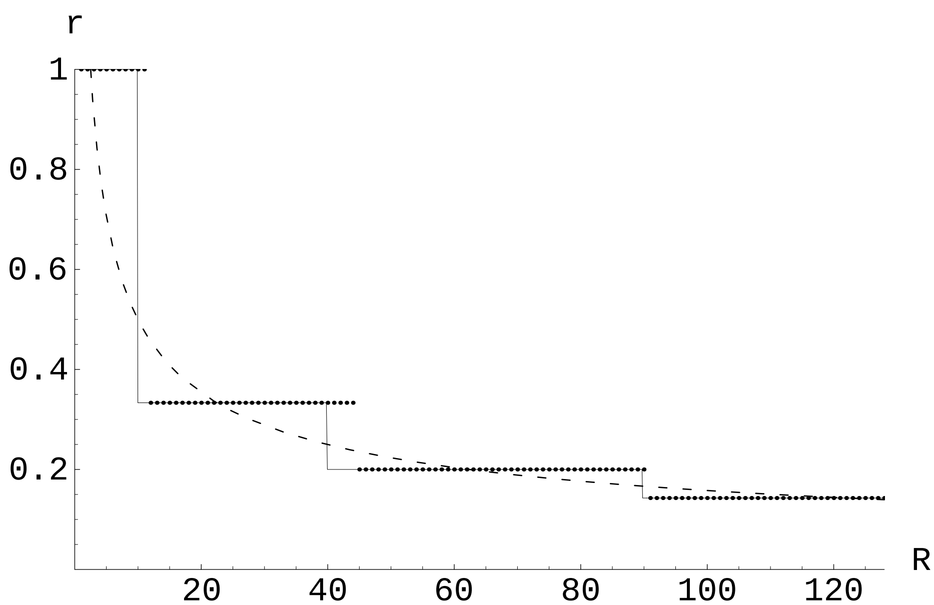

Figure 1: An illustration of Theorem 1 for uniform scalar quantization of a uniform source on $(0, 1)$ using repetition codes to transmit on a binary symmetric channel with $\epsilon = 10^{-3}$. The distortion minimizing channel code rate r is plotted against the overall transmission rate R.

Instead, the number of "dB's per bit" of performance gain in the bound (12) diminishes as the rate R grows. For example, increasing the total transmission rate R by 1 bit per component for a cascaded system using repetition codes yields a signal-to-noise ratio increase of

$$\begin{aligned} \mathrm{SNR}(R+1) - \mathrm{SNR}(R) &= 10 \log_{10} \left(2^{-2\sqrt{R}+O(1)} / 2^{-2\sqrt{R+1}+O(1)} \right) \\ &\approx \frac{3}{\sqrt{R}} \text{ [dB]}. \end{aligned}$$

However, the bounds presented might be improved in the future.

4 Conclusion

The paper presented bounds on the performance of implementable communication systems as a function of the overall transmission rate R. The systems employ a binary lattice vector quantizer for source coding a bounded random input, and a binary linear channel code for transmission over a binary symmetric channel. The channel code is obtained as a Cartesian product of short codes from channel code families with vanishing rate. Many well studied $[n, k]$ linear channel codes have k proportional to some power of $\log_2 n$. We showed that for such codes, using a rate allocation between source and channel coding of $O(\sqrt{\frac{\log_2^l R}{R}})$ as $R \to \infty$, one gets an asymptotic distortion decay of $2^{-2\sqrt{R \log_2^l R}}$. Since the exponent is sub-linear in R, we see diminishing returns in the per-bit performance increase instead of the usual 6 dB/bit for error-free transmission (or some other constant return for optimal or asymptotically good codes).

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