Designing Vector Quantizers in the Presence of Source Noise or Channel Noise *

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1 Abstract

The problem of vector quantizer empirical design for noisy channels or for noisy sources is studied. It is shown that the average squared distortion of a vector quantizer designed optimally from observing clean i.i.d. training vectors converges in expectation, as the training set size grows, to the minimum possible mean-squared error obtainable for quantizing the clean source and transmitting across a discrete memoryless noisy channel. Similarly, it is shown that if the source is corrupted by additive noise, then the average squared distortion of a vector quantizer designed optimally from observing i.i.d. noisy training vectors converges in expectation, as the training set size grows, to the minimum possible mean-squared error obtainable for quantizing the noisy source and transmitting across a noiseless channel. Rates of convergence are also provided.

2 Introduction

The design of quantizers has been studied over the last four decades from various perspectives. On the practical side, the Lloyd-Max [1], [2] algorithm provides an efficient iterative method of designing locally optimal quantizers from known source statistics or from training samples. The generalized Lloyd algorithm [3], [4] similarly is useful for designing vector quantizers. A theoretical problem motivated by practice is the question of consistency: if the observed training set size is large enough, can one expect a performance nearly as good as in the case of known source

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statistics? The consistency of design based on global minimization of the empirical distortion was established with various levels of generality by Pollard [5], Abaya and Wise [6], and Sabin [7]. The finite sample performance was also analyzed by Pollard [8], Linder, Lugosi, and Zeger [9], and Chou [10]. The consistency of the generalized Lloyd algorithm was also established by Sabin [7] and Sabin and Gray [12]. An interesting interpretation of the quantizer design problem was given by Merhav and Ziv [11], who obtained lower bounds on the amount of side information a quantizer design algorithm needs to perform nearly optimally for all sources.

Less is known about the more general situation when the quantized source is to be transmitted through a noisy channel (joint source and channel coding), or when the source is corrupted by noise prior to quantization (quantization of a noisy source). In the noisy channel case, theoretical research has mostly concentrated on the questions of optimal rate-distortion performance in the limit of large blocklength either for separate [13], or joint [14] source and channel coding, as well as for high resolution source-channel coding [15],[16]. Practical algorithms have also been proposed to iteratively design (locally) optimal source and channel coding schemes [17], [18].

For the noisy source quantization problem the optimal rate-distortion performance was analyzed by Dobrushin and Tsybakov [19] and Berger [20]. The structure of the optimal noisy source quantizer for squared distortion was studied by Fine [21], Sakrison [22], and Wolf and Ziv [23]. The framework of these works also included transmission through a noisy channel. Properties of optimal noisy source quantizers as well as a treatment of Gaussian sources corrupted by additive independent Gaussian noise were given by Ayanoglu [24]. A Lloyd-Max type iterative design algorithm was given by Ephraim and Gray [25] for the design of vector quantizers for noisy sources. A design approach based on deterministic annealing was reported by Rao *et al.* [26]. No consistency results have yet been proved for empirical design of noisy channel or noisy source vector quantizers.

In empirical design of standard vector quantizers one can observe a finite number of independent samples of the source vector. The procedure chooses the quantizer which minimizes the average distortion over this data. One is interested in the expected distortion of the designed quantizer when it is used on a source which is independent of the training data. An empirical design procedure is called *consistent* if the expected distortion of the empirical quantizer approaches the distortion of the quantizer which is optimal for the source, as the size of the training data increases. If consistency is established, one can investigate the rate of convergence of the algorithm, i.e., how fast the expected distortion of the empirically optimal quantizer approaches the optimal distortion. Tight convergence rates have practical significance, since consistency alone gives no indication of the relationship between the resulting distortion and the size of the training data.

In this paper we investigate the consistency of vector quantizers obtained by global empirical error minimization for noisy channels and noisy sources. In both cases, the notion of empirical (sample) distortion is not as simple as in standard vector quantizer design. For noisy channels, the channel transition probabilities are assumed to be known, and the empirical distortion is defined as the expected value of a source symbol and its random reproduction, where the expectation is taken with respect to the channel. For sources corrupted by noise, the density of the noise is assumed to be known and the estimation-quantization structure (see, e.g., [23]) of the optimal quantizer is used. Here the sample distortion has no unique counterpart. Although a modified distortion measure can be introduced [25] which converts the problem into a standard quantization problem, this modified measure cannot directly be used since it is a function of the unknown source statistics. The main difficulty lies in the fact that, in general, the encoding regions of a noisy source vector quantizer need not be either convex or connected. Thus the set of quantizers to be considered in the minimization procedure is more complex than in the clean source or noisy channel case.

3 Preliminaries

3.1 Vector quantizers for noisy channels

An N*-level noisy-channel vector quantizer* is defined via two mappings. The *encoder* Q_C maps $\mathcal{R}^k$ into the finite set $\{1,\ldots,N\}$, and the *decoder* Q_D maps $\{1,\ldots,N\}$ onto the set of codewords $\{y_1,y_2,\ldots,y_N\} \subset \mathcal{R}^k$ by the rule $Q_D(j) = y_j$, for $j = 1,\ldots,N$. The rate of the quantizer is $(1/k)\log N$ bits per source symbol. The quantizer takes an $\mathcal{R}^k$-valued random vector X as its input, and produces the index $I = Q_C(X)$. The index I is then transmitted through a noisy channel, and the decoder receives the index $J \in \{1,\ldots,N\}$, a random variable whose conditional distribution given I is

$$\mathbf{P}(J=j|I=i) = p(j|i), \quad 1 \le i,j \le N,$$

where the $p(j|i)$ are the channel transition probabilities. The channel is assumed to be discrete with N input and N output symbols, with known transition probabilities, and the channel is assumed to work independently of the source X. The output of the quantizer is

$$Y = Q_D(J) = y_J,$$

and the joint distribution of (X,Y) is determined by the source distribution and the conditional distribution

$$\mathbf{P}(Y=y_j|X=x) = p(j|Q_C(x)).$$

We will use the notation $Y = Q(X)$ as for an ordinary vector quantizer, but now Q is not a deterministic mapping. The quantizer distortion can be written as

$$\begin{aligned}\mathbf{E}[\,\|Q(X)-X\|^2\,] &= \mathbf{E}[\,\mathbf{E}[\|Q(X)-X\|^2\,|\,I,J]\,] \\ &= \sum_{i=1}^{N}\int_{R_i}\left(\sum_{j=1}^{N}\|y_j-x\|^2 p(j|i)\right)P_X(dx), \qquad (1)\end{aligned}$$

where the encoding regions $R_i = \{x : Q_C(x) = i\}$, for $i = 1, \dots N$ completely determine the encoder Q_C. It is obvious from (1) that given the decoder Q_D, the encoder regions

$$R_i = \left\{ x : \sum_{j=1}^{N} \|y_j - x\|^2 p(j|i) \le \sum_{j=1}^{N} \|y_j - x\|^2 p(j|l),\ l = 1, \dots, N \right\}$$

determine an encoder (with ties broken arbitrarily) which minimizes the distortion over all encoders. The above encoding rule is sometimes called the *weighted nearest neighbor condition* (see, e.g., [27], [17], [28], [14]).

Let us denote the distortion of an optimal quantizer Q_N^* by

$$D_N^* = \mathbf{E}[\, \|Q_N^*(X) - X\|^2\,] = \min_{Q_C, Q_D} \mathbf{E}[\|Q(X) - X\|^2],$$

where the minimum is taken over all (N-level) encoders and decoders operating on the fixed channel and source X.

3.2 Vector quantizers for noisy sources

Assume that Y is the noisy version of the source X. Y can be viewed as the output of a channel whose input is X. The noisy source Y is to be quantized by an N-level quantizer Q such that the mean squared distortion

$$\mathbf{E}[\, \|X - Q(Y)\|^2\,]$$

is as small as possible. In this problem an N-level quantizer Q is characterized by its codevectors $\{y_1, \dots, y_N\} \subset \mathcal{R}^k$ and the measurable sets $R_i = \{x \in \mathcal{R}^k : Q(x) = y_i\}$, $i = 1 \dots, N$, called *encoding regions.* As was noted in several papers dealing with this problem (see, e.g., [19], [21], [22], and [23]), the structure of the optimal N-level quantizer can be obtained via a useful decomposition. Let $M : \mathcal{R}^k \to \mathcal{R}^k$ denote a version of the conditional expectation $\mathbf{E}[X \,|\, Y = y\,]$. Then

$$\begin{aligned} \mathbf{E}[\, \|X - Q(Y)\|^2\,] &= \mathbf{E}[\, \|X - M(Y)\|^2\,] + \mathbf{E}[\, \|M(Y) - Q(Y)\|^2\,] \\ &\quad + 2\mathbf{E}[(X - M(Y))^t(M(Y) - Q(Y))\,] \\ &= \mathbf{E}[\, \|X - M(Y)\|^2\,] + \mathbf{E}[\, \|M(Y) - Q(Y)\|^2\,] \qquad (2) \end{aligned}$$

where the cross-term disappears after taking iterated expectations, first conditioned on Y. If the codevectors $\{y_1, \dots, y_N\}$ are given, then the encoding regions minimizing the distortion must satisfy

$$\|M(y) - y_i\| \le \|M(y) - y_j\|, \text{ for } \quad j = 1, \dots, N \;\text{ if } y \in R_i. \qquad (3)$$

This means that for any Q,

$$\mathbf{E}[\, \|M(Y) - Q(Y)\|^2\,] \le \mathbf{E}[\, \|M(Y) - \hat{Q}(M(Y))\|^2\,],$$

where $\widehat{Q}$ is an ordinary *nearest neighbor* quantizer which has the same codevectors as Q. Thus by (2) we have

$$\begin{aligned} D_N^* &\stackrel{\text{def}}{=} \inf_Q \mathbf{E}[\,\|X - Q(Y)\|^2\,] \\ &= \mathbf{E}[\,\|X - M(Y)\|^2\,] + \inf_{\widehat{Q}} \mathbf{E}[\,\|M(Y) - \widehat{Q}(M(Y))\|^2\,], \end{aligned}$$

where the second infimum is taken over all N-level nearest neighbor quantizers $\widehat{Q}$. The quantizer Q^* minimizing $\mathbf{E}[\,\|X - Q(Y)\|^2\,]$ is obtained by first transforming Y by M and then quantizing $M(Y)$ by a nearest neighbor quantizer $\widehat{Q}^*$, that is,

$$Q^*(Y) = \widehat{Q}^*(M(Y)).$$

Furthermore,

$$D_N^* = \mathbf{E}[\,\|X - M(Y)\|^2\,] + \mathbf{E}[\,\|M(Y) - \widehat{Q}^*(M(Y))\|^2\,]. \tag{4}$$

4 Empirical Design for Noisy Channels

In most applications one does not know the actual source statistics, but instead can observe a sequence of i.i.d. copies $Z_m = (X_1, X_2, \ldots, X_m)$ of X. These m "training samples" induce the empirical distribution P_m which assigns probability to every measurable $G \subset \mathcal{R}^k$ according to the rule

$$P_m(G) = \frac{1}{m}\sum_{l=1}^{m} I_{\{X_l \in G\}},$$

where I is the indicator function of the event of its argument. When the source statistics are not known, one cannot directly search for an optimal quantizer Q^*. Instead, one generally attempts to minimize the empirical distortion, which is a functional of P_m rather than of the true source distribution. The empirical distortion $D_{N,m}$ is the expected value (expectation taken over the channel use) of the average distortion of the quantizer when Z_m is quantized,

$$D_{N,m} = \sum_{i=1}^{N} \int_{R_i} \left(\sum_{j=1}^{N} \|y_j - x\|^2 p(j\,|\,i) \right) P_m(dx). \tag{5}$$

The empirical distortion can be rewritten in the simple form

$$D_{N,m} = \frac{1}{m}\sum_{l=1}^{m} d_Q(X_l),$$

where $d_Q : \mathcal{R}^k \to \mathcal{R}^+$ is a function which depends on the quantizer Q as

$$d_Q(x) = \sum_{i=1}^{N} I_{\{x \in R_i\}} \left(\sum_{j=1}^{N} \|y_j - x\|^2 p(j\,|\,i) \right). \tag{6}$$

Assume we design a quantizer based on the training data by *minimizing the empirical distortion* over all possible quantizers. This minimization can be carried out in principle, since given Z_m and the channel transition probabilities, we can calculate $D_{N,m}$ for any quantizer using weighted nearest neighbor encoding.

Let $Q_N^*(\cdot|Z_m)$ be the quantizer minimizing $D_{N,m}$,

$$Q_N^*(\cdot|Z_m) = \arg\min_Q \frac{1}{m}\sum_{l=1}^m d_Q(X_l),$$

and let

$$D_{N,m}^* = \mathbf{E}[\|Q_N^*(X|Z_m) - X\|^2],$$

where X is independent of Z_m. Then $D_{N,m}^*$ is the average distortion of the empirically optimal quantizer when it is used on data independent of the training set. A fundamental question is how close this distortion gets to the optimal D_N^* as the size of the training data increases, and therefore as the source statistics are more and more revealed by the empirical distribution.

One goal in this paper is to investigate how fast the difference between the expected distortion of the empirically optimal quantizer and the optimal distortion

$$\mathbf{E}[\,\|Q_N^*(X|Z_m) - X\|^2\,] - D_N^*$$

decreases as the training set size m increases.

Theorem 1 *Assume that a source $X \in \mathcal{R}^k$ is bounded as $\mathbf{P}(\|X\|^2 \le B) = 1$ for some $B > 0$, and let $Z_m = (X_1, \ldots, X_m)$, where the X_i are i.i.d. copies of X. Suppose an N-level noisy channel vector quantizer $Q_N^*(\cdot|Z_m)$ is designed by using empirical distortion minimization over the training set Z_m. Then the average distortion of this quantizer is bounded above as*

$$\mathbf{E}[\,\|Q_N^*(X|Z_m) - X\|^2\,] \le D_N^* + c\sqrt{\frac{\log m}{m}} + O(m^{-1/2}),$$

where D_N^* is the distortion of the N-level quantizer that is optimal for the source and the channel, and $c = 8B\sqrt{kN+1}$.

5 Empirical Design for Noisy Source

In the noisy source quantizer design problem we are given the samples $Z_m = (Y_1, \ldots, Y_m)$ drawn independently from the distribution of Y. We also assume that the conditional distribution of the noisy source Y given X is known (i.e., the channel between X and Y is known), and that $\mathbf{P}(\|X\|^2 \le B) = 1$ for some *known* constant B. In this situation the method of empirical distortion minimization cannot be applied directly, since we only have the indirect (noisy) observations $Y_1, \ldots, Y_n$ about X. However, the decomposition (4) suggests the following method for noisy source quantizer design:

(i) Split the data Z_m into two parts, $Z_m^{(1)} = (Y_1, \ldots, Y_{m/2})$ and $Z_m^{(2)} = (Y_{m/2+1}, \ldots, Y_m)$ (assume m is even) and estimate $M(y) = \mathbf{E}[X|Y = y]$ from the first half of the samples $Z_m^{(1)}$ and the *known* conditional distribution $P_{X|Y}$. The estimate $M_m(\cdot) = M_m(\cdot, Z_m^{(1)})$ is required to be L_2 consistent:

$$\mathbf{E}[\,\|M_m(Y) - M(Y)\|^2\,] = a_m \to 0 \quad \text{as} \quad m \to \infty. \tag{7}$$

Since the upper bound B on $\|X\|^2$ is known we also require that

$$\sup_{y \in \mathcal{R}^k} \|M_m(y)\|^2 \le B. \tag{8}$$

(ii) Using the second half of the training data define a new set of $m/2$ training vectors $M_m(Y_{m/2+1}), \ldots, M_m(Y_m)$, and consider a nearest neighbor quantizer $\hat{Q}_m^*$ minimizing the empirical distortion:

$$\hat{Q}_m^* = \arg\min_Q \frac{1}{m/2} \sum_{i=m/2+1}^{m} \|M_m(Y_i) - Q(M_m(Y_i))\|^2. \tag{9}$$

Here the minimization is over all N-level nearest neighbor quantizers. The quantizer for the noisy source designed from the noisy samples is then obtained from $\hat{Q}_m^*$ and M_m as

$$Q_m^* = \hat{Q}_m^* \circ M_m.$$

Theorem 2 *Assume that a source $X \in \mathcal{R}^k$ is bounded as $\mathbf{P}(\|X\|^2 \le B) = 1$ for some $B > 0$ and let $(Y_1, \ldots, Y_m)$ be i.i.d. samples of the noisy source Y. Suppose furthermore that the conditional distribution of Y given X, and the constant B are known, and that the estimator $M_m(y)$ of $M(y) = \mathbf{E}[\,X\,|\,Y = y]$ has L_2 error*

$$\mathbf{E}[\,\|M_m(Y) - M(Y)\|^2\,] = a_m,$$

and is bounded as

$$\sup_{y \in \mathcal{R}^k} \|M_m(y)\|^2 \le B.$$

Then the N-level Q^* quantizer designed in steps (i) *and* (ii) *above satisfies*

$$\mathbf{E}[\,\|X - Q_m^*(Y)\|^2\,] \le D_N^* + c\sqrt{\frac{\log m}{m}} + O(m^{-1/2}) + 8\sqrt{B a_m} + a_m$$

where D_N^* is the distortion of the optimal N-level quantizer for the noisy source problem, and $c = 8B\sqrt{2(kN+1)}$.

Corollary 1 *Assume the conditions of Theorem 2 and suppose $Y = X + \nu$, where ν is independent of X and has a density whose characteristic function is almost everywhere nonzero. Then there exists a bounded estimator M_m of M such that*

$$\lim_{m \to \infty} \mathbf{E}[\,\|M(Y) - M_m(Y)\|^2\,] = 0,$$

and the noisy source design procedure is consistent, i.e.,

$$\lim_{m \to \infty} \mathbf{E}[\,\|X - Q_m^*(Y)\|^2\,] = D_N^*.$$

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