# Universality and Rates of Convergence in Lossy Source Coding *


Tamás Linder
Faculty of Elect. Engineering
Technical University of Budapest
1521 Stoczek u. 2, Budapest, Hungary

Gábor Lugosi
Dept. of Mathematics
Technical University of Budapest
1521 Stoczek u. 2, Budapest, Hungary

Kenneth Zeger
Coordinated Science Laboratory
Dept. of Elect. and Comp. Engineering
University of Illinois, Urbana-Champaign, IL 61801


## 1 Abstract


We show that without knowing anything about the statistics of a bounded, real-valued memoryless source, it is possible to construct a sequence of codes, of rate not exceeding a fixed number $R > 0$, such that the per letter sample distortion converges to the distortion-rate function $D(R)$ with probability one as the length of the message approaches infinity. In addition, it is proven that the distortion converges to $D(R)$ as $\sqrt{\log\log n / \log n}$ almost surely, where $n$ is the length of the data to be transmitted.


## 2 Introduction

The problem of universal lossy coding, that is, transmitting messages from an unknown source under a certain rate $R$ was first formulated by Ziv [8], who considered a scheme whose expected distortion is proven to converge to $D(R)$ for any stationary, ergodic process. Problems of this type were further pursued by e.g. Neuhoff, Gray and Davisson [4], and Pursley and Davisson [7].

The coding scheme that we consider here is essentially Ziv's, which can be described as follows. Consider a real-valued data sequence containing $n$ samples from a source. First we parse the sequence into blocks of equal length $k_n$, and then


*The research was supported in part by the National Science Foundation under Grant No. NCR-91-57770.

design a $k_n$-dimensional vector quantizer with $N_n$ reproduction points (codevectors) that gives the best performance on the vectors obtained from the data. Then we uniformly quantize the coordinates of the $N_n$ quantizer codevectors using $b_n$ bits per coordinate and then transmit this approximation of the codebook. Finally, we use the quantized quantizer to transmit approximations to the $k_n$-long source vectors. For a given input sequence length $n$, the choice of the block length $k_n$ determines both the vector quantizer dimension (i.e. $k_n$) and the size (i.e. $n/k_n$) of the vector training set used to design the vector quantizer. The problem is to choose the parameters of the code $k_n, N_n$, and $b_n$ such that the overall per-letter transmission rate stays below a fixed number $R$, and the per-letter distortion approaches $D(R)$ almost surely. We will show first that such a choice possible, and second (and perhaps more surprising) that by properly choosing the parameters there is a *universal rate of convergence* that applies for *all* memoryless sources with bounded support.

To establish such a result, on the one hand, we have to study the rate of convergence of the emirical performance of the empirically designed $k_n$-dimensional vector quantizer. The other side of the trade-off is the difference between the distortion of the best $k_n$-dimensional quantizer of rate not exceeding $R$ and the distortion-rate function $D(R)$. Shannon's classical source coding theorem tells us that the difference always goes to zero so long as the dimension $k_n$ grows to infinity. This fact is sufficient for the existence of universal coding schemes. However, to derive conditions on how one should optimally choose the quantizer dimension, one needs to study the rate at which this difference goes to zero. In Section 5 we discuss the problem for memoryless real sources.

# 3 Universal source coding

The distortion rate function $D(R)$ with respect to the mean squared distortion of a real i.i.d. source $X_1, X_2, \ldots$ is defined for $R \geq 0$ as

$$D(R) = \inf_Y E|X - Y|^2,$$

where $X$ has the common distribution of the $X_i$'s and the infimum is taken over all real random variables $Y$ such that the mutual information between $X$ and $Y$ is at most $R$:

$$I(X;Y) \leq R.$$

Since we are dealing with arbitrary real variables, the definition of $I(X;Y)$ cannot be given in terms of the joint density of $(X,Y)$; a more general definition [1] is needed. Let $P_{XY}$ denote the probability measure induced by the pair $(X,Y)$ and let $P_{X\times Y}$ be the product of the marginals $P_X$ and $P_Y$ of $P_{XY}$. If $P_{XY}$ is absolutely continuous with respect to $P_{X\times Y}$ with Radon-Nikodym derivative $a(x,y)$, we have

$$I(X;Y) = \int \log a(x,y) P_{XY}(dx,dy),$$

otherwise $I(X;Y) = \infty$. If $X$ has a finite second moment, $D(R) < \infty$ for all $R \geq 0$. In the sequel we assume that $D(R) > 0$ (which is satisfied for all $R \geq 0$ if the distribution of $X$ has a continuous component). Next we introduce the notion of universal lossy coding.

**Definition 1** *A sequence of pairs of functions* $(f_n, \phi_n)$ *of the form*

$$f_n : \mathcal{R}^n \to \{0,1\}^{\lfloor nR \rfloor} \quad and \quad \phi_n : \{0,1\}^{\lfloor nR \rfloor} \to \mathcal{R}^n$$

*is called a* **universal source coding scheme** *with respect to a family of sources if the random vector* $(Y_{1,n}, \ldots, Y_{n,n}) = \phi_n(f_n(X_1, \ldots, X_n))$ *satisifies*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} (X_i - Y_{i,n})^2 = D(R)$$

*almost surely (a.s.) for all possible sources in the family.*

Ziv [8] showed that there exists a sequence of pairs of functions $(f_n, \phi_n)$ such that

$$\lim_{n\to\infty} E\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - Y_{i,n})^2\right) = D(R) \tag{1}$$

for the family of all ergodic sources. The main purpose of the present paper is to strengthen Ziv's result from convergence in mean to almost sure convergence, and to establish a universal rate-of-convergence result. We do this for the class of memoryless sources with bounded support, that is, for i.i.d. sources such that there exists a bounded set $K \subset \mathcal{R}$ such that $\Pr\{X_1 \in K\} = 1$.

**Theorem 1** *For every* $R > 0$ *there exists a universal source coding scheme for the family of i.i.d. real sources with bounded support. Moreover, if* $D(R)$ *is the distortion-rate function of any such source, then for every* $R > 0$ *such that* $D(R) > 0$ *there exists a constant* $c$ *such that for every* $n$ *the difference between the empirical distortion and* $D(R)$ *satisfies*

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - Y_{i,n})^2 - D(R) \le c\sqrt{\log\log n/\log n} + o\left(\sqrt{(\log n/n)^{1-\epsilon}}\right) \qquad a.s.$$

*for any* $0 < \epsilon < 1$.

Theorem 1 will be proved in Section 6.

# 4 Designing quantizers from data

Let $Z, Z_1, \ldots, Z_m$ be independent, identically distributed random variables, taking their values from $\mathcal{R}^k$. A quantizer (or *vector quantizer*) is a measurable function of the form $Q_{N,k} : \mathcal{R}^k \to \{y_1, \ldots, y_N\} \subset \mathcal{R}^k$ such that

$$Q_{N,k}(z) = y_i \text{ if } \|x - y_i\|^2 < \|x - y_j\|^2 \text{ for all } j \neq i.$$

Note, that a quantizer is determined by its range $\{y_1, \ldots, y_N\}$ via the nearest neighbor quantization rule. Given a quantizer $Q_{N,k}$ define its *average distortion* as

$$\Delta(Q_{N,k}) = E\|Z - Q_{N,k}(Z)\|^2$$

and *empirical distortion* as

$$\Delta_m(Q_{N,k}) = \frac{1}{m}\sum_{i=1}^{m} \|Z_i - Q_{N,k}(Z_i)\|^2.$$

$\Delta(Q_{N,k})$ is the average distortion one obtains by designing a vector quantizer based on training data and then measuring its performance on a true source, whereas $\Delta_m(Q_{N,k})$ (a random variable since it depends on $Z_1, \ldots, Z_m$) is the distortion obtained by designing a vector quantizer based on a training set and then using the resulting quantizer to encode the same training set.

Let $Q^*_{N,k}$ be a quantizer of minimal average distortion, and $Q^*_{m,N,k}$ a quantizer with minimal empirical distortion. That is,

$$Q^*_{N,k} = \arg\min_{Q_{N,k}} \Delta(Q_{N,k})$$

and

$$Q^*_{m,N,k} = \arg\min_{Q_{N,k}} \Delta_m(Q_{N,k}).$$

We are interested in the following random variable:

$$\Delta_m(Q^*_{m,N,k}) - \Delta(Q^*_{N,k}), \tag{2}$$

that is, in the difference between the empirical distortion of the empirically optimal quantizer $Q^*_{m,N,k}$ and the (expected) distortion of the (truly) optimal quantizer $Q^*_{N,k}$. This difference quantifies the gap between the best theoretically achievable performance and the best performance achievable in coding the training set. Observe that the expected value of the random variable in (2) is always less than or equal to zero, since

$$E\Delta_m(Q^*_{m,N,k}) \leq E\Delta_m(Q^*_{N,k}) = \Delta(Q^*_{N,k}). \tag{3}$$

Next we show that if the random variables $Z_1, \ldots, Z_m$ are independent and identically distributed (i.i.d.) on a bounded subset of $\mathcal{R}^k$, then the probability that $\Delta_m(Q^*_{m,N,k})$ is larger than $\Delta(Q^*_{N,k})$ plus any positive constant is exponentially small in $m$.

**Lemma 1** *Assume that $Z_1, Z_2, \ldots$ form an i.i.d. sequence and there exists a bounded set $A \subset \mathcal{R}^k$ such that $\Pr\{Z_1 \in A\} = 1$. Then for every $t > 0$ the difference between the empirical distortion of the empirically optimal quantizer and the distortion of the best possible quantizer satisfies*

$$\Pr\{\Delta_m(Q^*_{m,N,k}) - \Delta(Q^*_{N,k}) > t\} \leq e^{-2mt^2/B^2},$$

*where the constant $B$ is a number such that $\|x\|^2 \leq B$ if $x \in A$.*

**Proof of Lemma 1** Clearly,

$$\begin{aligned}\Pr\{\Delta_m(Q^*_{m,N,k}) - \Delta(Q^*_{N,k}) > t\} &\leq \Pr\{\Delta_m(Q^*_{N,k}) - \Delta(Q^*_{N,k}) > t\} \\ &= \Pr\left\{\frac{1}{m}\sum_{i=1}^{m}\|Z_i - Q^*_{N,k}(Z_i)\|^2 - E\|Z - Q^*_{N,k}(Z)\|^2 > t\right\} \\ &\leq 2e^{-2mt^2/B^2},\end{aligned}$$

by Hoeffding's inequality [3], where $B$ is a bound on $\|Z - Q^*_{N,k}(Z)\|^2$. □

# 5 Rate of convergence in the source coding theorem

In this section we demonstrate an upper bound on the rate at which the distortion $D_k(R)$ of the best $k$-dimensional quantizer of rate $R$ converges to the distortion rate function as $k \to \infty$. For finite alphabet i.i.d. sources the problem was settled by Pilc [5] who showed upper and lower bounds of the type $c \log k/k$ on the difference $D_k(R) - D(R)$, where $D(R)$ is the distortion-rate function.

Although it seems possible to extend Pilc's result to bounded continuous alphabets, the argument would be unduly lengthy. Thus we settle for extending Gallager's $O\left(\sqrt{\log k/k}\right)$ rate of convergence result [2]. We consider only mean squared distortion; the result straightforwardly extends to distortion measures $\rho : \mathcal{R} \times \mathcal{R} \to \mathcal{R}^+$ such that $\rho(x,x) = 0$ and $\rho$ is bounded over compact subsets of $\mathcal{R}^2$.

Let $X^{(k)} = (X_1, \ldots, X_k)$. If $E|X|^2 < \infty$, the source coding theorem for memoryless sources [2, Theorem 9.6.2] asserts that

$$\begin{aligned}D(R) &= \lim_{k\to\infty}\left\{\inf_{Q_{N,k}:\frac{1}{k}\log N \leq R}\left\{\frac{1}{k}E\|Q_{N,k}(X^{(k)}) - X^{(k)}\|^2\right\}\right\} \\ &= \lim_{k\to\infty} D_k(R),\end{aligned}$$

where the infimum is taken over all $k$-dimensional $N$-point quantizers $Q_{N,k}$ with rate $\frac{1}{k}\log N$ at most $R$. We note that in this situation an optimal quantizer achieving this minimum always exists [6]. When $|X| \leq c_0 < \infty$ a.s., an upper bound on the rate of this convergence is given by

**Theorem 2** *Let $X$ be a scalar source that is bounded with probability one. Let $D_k(R)$ be the minimum mean squared distortion of any $k$-dimensional vector quantizer for $X$ and let $D(R)$ be the distortion-rate function of $X$. Then for every $R \geq 0$ there is a constant $c$ such that for every $k$*

$$D_k(R) - D(R) \leq c\sqrt{\frac{\log k}{k}}.$$

The proof of Theorem 2 will appear in an upcoming publication.

# 6 Universal source coding schemes

In this section we prove Theorem 1 by constructing a universal source coding scheme. In the sequel we will omit the subscripts from $b_n, k_n, N_n$ and $m_n$, but they always will be understood as functions of $n$. Now, we are prepared to describe the universal schemes that provide constructive proofs for Theorem 1. Split the data $X^{(n)}$ into $k$-long blocks $Z_1, \ldots, Z_m$ ($m = \lfloor \frac{n}{k} \rfloor$), where $Z_i = (X_{(k-1)i+1}, \ldots, X_{ki})$ is the $k$-dimensional vector formed by the $i$-th block of $X^{(n)}$. Take the $k$-dimensional quantizer with $N$ codevectors that minimizes the empirical distortion over $Z_1, \ldots, Z_m$:

$$Q^*_{m,N,k} = \arg\min_{Q_{N,k}} \frac{1}{m} \sum_{i=1}^{m} \|Z_i - Q_{N,k}(Z_i)\|^2.$$

Quantize uniformly each of the coordinates of the codevectors of $Q^*_{m,N,k}$ with $b$ bits, and transmit them. Denote the quantized quantizer by $\hat{Q}^*_{m,N,k}$, that is, $\hat{Q}^*_{m,N,k}$ is the quantizer whose codevectors are the codevectors of $Q^*_{m,N,k}$ quantized uniformly using $b$ bits per coordinate. Finally, quantize the vectors $Z_i$ with $\hat{Q}^*_{m,N,k}$, and transmit them. The remaining letters of $X^{(n)}$ are not transmitted, and will be decoded as zeros. Observe, that to transmit the quantizer, we use $bkN$ bits, since every coordinate of the $N$ codevectors is quantized with $b$ bits. To transmit the $Z_i$'s we use $m \log N$ bits. Therefore, the overall per letter rate is

$$r_n = \frac{m \log N}{n} + \frac{bkN}{n}. \tag{4}$$

The reproduction value for a block $Z_i = (X_{(k-1)i+1}, \ldots, X_{ki})$ will be the codevector of the quantized version of the $Q^*_{m,N,k}$ which is closest to $Z_i$, and zero for $X_{mk+1}, \ldots, X_n$. Formally,

$$(Y_{(k-1)i+1,n}, \ldots, Y_{ki,n}) = \hat{Q}^*_{m,N,k}(Z_i) \quad \text{for} \quad i = 1, \ldots, m,$$

and

$$Y_j = 0 \quad \text{for} \quad j = mk+1, \ldots, n.$$

Our aim is to choose the parameters $k, N$ and $b$ (as functions of $n$) such that $r_n \leq R$ and the overall per letter distortion converges to $D(R)$.

**Proof of Theorem 1** Assume for the sake of simplicity, that a positive constant $c_0$ is known to the receiver for which $\Pr\{|X_1| \leq c_0\} = 1$. Otherwise a bound for $\max(|X_1|, \ldots, |X_n|)$ can be transmitted using a constant number of bits, and therefore, will not effect the rate asymptotically. The overall per-letter sample distortion is

$$\begin{aligned}
\frac{1}{n} \sum_{i=1}^{n} (X_i - Y_{i,n})^2 &= \frac{1}{mk} \sum_{j=1}^{m} \|Z_j - \hat{Q}^*_{m,N,k}(Z_j)\|^2 + \frac{1}{n} \sum_{i=km+1}^{n} |X_i|^2 \\
&= \frac{1}{k} \Delta_m(\hat{Q}^*_{m,N,k}) + \frac{1}{n} \sum_{i=km+1}^{n} |X_i|^2.
\end{aligned}$$

We want to show that there is a choice of the parameters $k, N$ and $b$ such that

$$\sqrt{\frac{\log n}{\log\log n}}\left(\frac{1}{n}\sum_{i=1}^{n}(X_i - Y_{i,n})^2 - D(R)\right)$$

remains bounded almost surely, while $r_n \leq R$. In order to do so, we use the following decomposition:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n}(X_i - Y_{i,n})^2 - D(R) &= \frac{1}{n}\sum_{i=km+1}^{n}|X_i|^2 \\
&\quad + \left(\frac{1}{k}\Delta_m(\hat{Q}^*_{m,N,k}) - \frac{1}{k}\Delta_m(Q^*_{m,N,k})\right) \\
&\quad + \left(\frac{1}{k}\Delta_m(Q^*_{m,N,k}) - \frac{1}{k}\Delta(Q^*_{N,k})\right) \\
&\quad + \left(\frac{1}{k}\Delta(Q^*_{N,k}) - D(R)\right). \qquad (5)
\end{aligned}$$

In the sequel we give upper bounds for the four terms on the right hand side of (5). For the first term, by the boundedness assumption obviously

$$\frac{1}{n}\sum_{i=km+1}^{n}\|X_i\|^2 \leq \frac{kc_0}{n} \quad \text{a.s..} \qquad (6)$$

Therefore, $\frac{1}{n}\sum_{i=km+1}^{n}\|X_i\|^2 \to 0$ a.s, if

$$k/n \to 0. \qquad (7)$$

For the second term observe first that by the triangle inequality

$$\begin{aligned}
\|Z - \hat{Q}^*_{m,N,k}(Z)\|^2 &= \|Z - Q^*_{m,N,k}(Z) + Q^*_{m,N,k}(Z) - \hat{Q}^*_{m,N,k}(Z)\|^2 \\
&\leq \|Z - Q^*_{m,N,k}(Z)\|^2 + 2\|Z - Q^*_{m,N,k}(Z)\|\|Q^*_{m,N,k}(Z) - \hat{Q}^*_{m,N,k}(Z)\| \\
&\quad + \|Q^*_{m,N,k}(Z) - \hat{Q}^*_{m,N,k}(Z)\|^2 \\
&\leq \|Z - Q^*_{m,N,k}(Z)\|^2 + 6c_0\sqrt{k}\|Q^*_{m,N,k}(Z) - \hat{Q}^*_{m,N,k}(Z)\| \quad \text{a.s.,}
\end{aligned}$$

where $c_0\sqrt{k}$ is a bound on $\|Z\|$ (a.s.). On the other hand, since each coordinate of the codevector of $Q^*_{m,N,k}$ is uniformly quantized with $b$ bits,

$$\|Q^*_{m,N,k}(Z) - \hat{Q}^*_{m,N,k}(Z)\| \leq \frac{c_0\sqrt{k}}{2^b} \quad \text{a.s.}$$

Therefore, we can bound the second term of (5) as

$$\begin{aligned}
&\frac{1}{k}\Delta_m(\hat{Q}^*_{m,N,k}) - \frac{1}{k}\Delta_m(Q^*_{m,N,k}) \\
&= \frac{1}{k}\left(\frac{1}{m}\sum_{j=1}^{m}\left(\|Z_j - \hat{Q}^*_{m,N,k}(Z_j)\|^2 - \|Z_j - Q^*_{m,N,k}(Z_j)\|^2\right)\right) \\
&\leq \frac{1}{km}\sum_{j=1}^{m}6c_0\sqrt{k}\frac{c_0\sqrt{k}}{2^b} = \frac{6c_0^2}{2^b} \quad \text{a.s.,} \qquad (8)
\end{aligned}$$

which goes to zero if $b \to \infty$ as $n \to \infty$. For the third term we have by Lemm
that for every $t > 0$

$$\Pr\left\{\frac{1}{k}\left(\Delta_m(Q^*_{m,N,k}) - \Delta(Q^*_{N,k})\right) > t\right\} \le e^{-2mk^2t^2/(kc_0^2)^2} = e^{-2mt^2/c_0^4}.$$

Therefore, $\sqrt{m^{1-\delta}}\frac{1}{k}\left(\Delta_m(Q^*_{m,N,k}) - \Delta(Q^*_{N,k})\right) \to 0$ a.s. for any $\delta > 0$ by the Bo
Cantelli lemma.
To handle the last term of (5), note, that by Theorem 2 and the convexity of $D($
there is a constant $c_2$ such that $\frac{1}{k}\Delta(Q^*_{N,k}) - D(R) \le c_2\left(\sqrt{\frac{\log k}{k}} + R - \frac{\log N}{k}\right)$.

Therefore, combining the bounds we have

$$\frac{1}{n}\sum_{i=1}^{n}(X_i - Y_{i,n})^2 \quad - \quad D(R)$$
$$\le \quad c\left(\sqrt{\frac{\log k}{k}} + 2^{-b} + R - \frac{\log N}{k}\right) + \frac{1}{k}\left(\Delta_m(Q^*_{m,N,k}) - \Delta(Q^*_{N}\right.$$

almost surely, for some constant $c$. Now, the parameters are to be chosen to minim
this upper bound, subject to the constraint

$$r_n = \frac{\log N}{k} + \frac{bkN}{n} \le R.$$

The choice

$$k_n = \frac{1}{R}(1-\epsilon)\log n, \quad N = \lfloor 2^{k(R-1/\log n)} \rfloor \quad b = \log\log n$$

gives the desired result.

# References


[1] T. Berger, "*Rate Distortion Theory*", Prentice–Hall, Englewood Cliffs, N
Jersey, 1971

[2] R. G. Gallager, "*Information Theory and Reliable Communication*," Wiley, N
York, 1968

[3] W. Hoeffding, "Probability inequalities for sums of bounded random variabl
*Journal of the American Statistical Association* 1963, 58, pp. 13–30,

[4] D. L. Neuhoff and R. M. Gray and L. D. Davisson, "Fixed rate universal bl
source coding with a fidelity criterion," *IEEE Trans. Inform. Theory* , 1
Sept., IT-21, pp.511–523

[5] R. Pilc, "The transmission distortion of a source as a function of the encod
block length," *Bell System Technical Journal*, 1968, 47, pp.827–885

D. Pollard, "Strong consistency of $k$-means clustering," *Annals of Statistics*, 1981, vol. 9, no. 1, pp.135–140

M. B. Pursley and L. D. Davisson, "Variable rate coding for nonergodic sources and classes of ergodic sources subject to a fidelity constraint," *IEEE Trans. Inform. Theory*, 1976, May, IT-22, pp.324–337

J. Ziv, "Coding of sources with unknown statistics- Part II: Distortion relative to a fidelity criterion," *IEEE Trans. Inform. Theory* , 1972, May, IT-18, pp.389–394