New Binary Fix-Free Codes with Kraft Sum $3/4$

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Abstract — **Two sufficient conditions are given for the existence of binary fix-free codes. The results move closer to the Ahlswede-Balkenhol-Khachatrian conjecture that Kraft sums of at most $3/4$ suffice for the existence of fix-free codes.**

For each nonnegative integer n let $\{0,1\}^n$ denote the set of all binary words of length n, and let $\{0,1\}^*$ denote the set of all finite length binary words, including the empty word. A binary *code* is any finite subset of $\{0,1\}^*$ that does not contain the empty word. The elements of a code are called *codewords*. For any two words $u, v \in \{0,1\}^*$, let uv denote the concatenation of u and v. The word u is called a *prefix* of uv and v is called a *suffix* of uv. A *prefix-free code* is a code such that no codeword is a prefix of any other codeword. A *suffix-free code* is a code such that no codeword is a suffix of any other codeword. A *fix-free code* is a code that is a both a prefix-free code and a suffix-free code. For any word $u \in \{0,1\}^*$, let $\ell(u)$ denote the length of u in bits, and let $\overline{u}$ denote the bitwise complement of u. If a set S of numbers if empty, then we adopt the convention $\max(S) = -\infty$.

In a fix-free code, any finite sequence of codewords can be decoded in both directions, which can improve robustness to channel noise.

For any nonnegative mapping $m : \mathbb{Z}^+ \longrightarrow \mathbb{Z}$, the *Kraft sum of* m is the quantity

$$S(m) = \sum_{i \in \mathbb{Z}^+} m(i) 2^{-i}.$$

If a code has exactly $m(i)$ codewords of length i for each $i \in \mathbb{Z}^+$, then we say the code is an m-code. The elements of the support $\mathrm{supp}(m) = \{i \in \mathbb{Z}^+ : m(i) > 0\}$ are called *lengths* and each quantity $m(i)$ is called the *multiplicity* of the length i. The mapping m is called a *multiplicity function*.

Kraft [2] showed in 1949 that every prefix-free code must have a Kraft sum of at most 1, and for every multiplicity function with Kraft sum at most 1, there exists a corresponding prefix-free code. The same result holds for suffix-free codes as well. Ahlswede, Balkenhol, and Khachatrian [1] conjectured in 1996 that an analogous result holds for fix-free codes, but with the Kraft sum bound being $3/4$ instead of 1. Specifically, they conjectured that if $S(m) \le 3/4$, then there exists a fix-free m-code.

They proved the conjecture is true in the weaker case when the Kraft sum is at most $1/2$. They also proved the converse of the conjecture, namely that any Kraft sum bound guaranteeing the existence of a fix-free code must be at most $3/4$. There are clearly fix-free codes whose Kraft sum is larger than $3/4$ (such as the set of all binary words of a given length, whose Kraft sum is 1), but these do not violate the conjecture. Instead, the conjecture gives the Kraft bound as a *sufficient* condition to guarantee the existence of a fix-free codes.

Ahlswede, Balkenhol, and Khachatrian proved their conjecture in the special case where every two codewords either have the same length or have one codeword at least twice as long as the other codeword. Since their conjecture was made, several researchers have proven other special cases, although the general conjecture still remains an open problem.

Harada and Kobayashi [3] showed that if $|\mathrm{supp}(m)| \le 2$ and $S(m) \le 3/4$, then there exists a fix-free m-code. Ye and Yeung [4] showed that if $1 \in \mathrm{supp}(m)$ and $S(m) \le 5/8$, then there exists a fix-free m-code. The also showed that if $\max(\mathrm{supp}(m)) \le 7$ and $S(m) \le 3/4$, then there exists a fix-free m-code. Yekhanin [5] showed that if $\max(\mathrm{supp}(m)) \le 8$ and $S(m) \le 3/4$, then there exists a fix-free m-code. Yekhanin also showed that if $1 \in \mathrm{supp}(m)$ and $S(m) \le 3/4$, then there exists a fix-free m-code. In addition, Ye and Yeung gave some other sufficient conditions for the conjecture to hold, although not in the form of Kraft sum bounds.

In this paper, we partly prove the conjecture by considering two special cases (Theorems 1 and 2). In both cases, we prove the conjecture holds if an additional constraint is put on the multiplicity function. We demonstrate that the classes of multiplicity functions for which our results hold contains many cases not covered by previous known results.

Theorem 1 *Let m be a multiplicity function such that*

$$\max\{m(i) : i \ne \max(\mathrm{supp}(m))\} \le 2^{\min(\mathrm{supp}(m))-2}.$$

If $S(m) \le 3/4$, then there exists a fix-free m-code.

Theorem 2 *Let m be a multiplicity function such that $m(i) \le 2$ for all $i \ne \max(\mathrm{supp}(m))$.*
If $S(m) \le 3/4$, then there exists a fix-free m-code.

References

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Supported in part by the National Science Foundation and by the Swiss National Science Foundation.