Cell Density Functions and Effective Channel Code Rates for Quantizers with Uniform Decoders and Channel Optimized Encoders

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I. Introduction

One approach to improving the performance of a quantizer that transmits across a noisy channel is to design the quantizer's encoder and/or decoder to take into account the statistics of the channel. Necessary optimality conditions for such channel optimized encoders and decoders were given in [3]. Alternatively, an explicit error control code can be cascaded with the quantizer, at the expense of added transmission rate. Additionally, the choice of index assignment in mapping source code words to channel code words can increase the performance of a quantization system with a noisy channel.

We examine quantizers with uniform decoders and channel optimized encoders. In particular, we investigate the Natural Binary Code index assignment and we introduce a new affine index assignment which we call the Modified Natural Code. We analyze the occurrence of empty encoding cells in such quantizers by computing their "effective channel code rate", which describes the number of empty cells, viewed in terms of implicit channel coding. We also examine the high resolution distribution of their encoding cells (i.e. the cell density function), and the mean squared errors the quantizers achieve.

II. Encoder Optimized Uniform Quantizers

The cascade of a rate k quantizer having no empty cells with a (n,k) block channel code can equivalently be viewed as a rate n quantizer with 2^n cells, 2^k of which are nonempty. To quantify the amount of natural error protection embedded in quantizers designed for noisy channels, we define the *effective channel code rate* of a quantizer as $r_c = (1/n)\log\left|\{\textit{nonempty encoding cells}\}\right|$. The effective channel code rate of a rate k quantizer cascaded with an (n,k) block channel code (viewed as a rate n quantizer) is k/n, i.e., the rate of the channel code. We compute r_c for certain quantizers that can not be decomposed as cascades of (lower transmission rate) quantizers with channel codes.

Let an *encoder optimized uniform quantizer* denote a rate n quantizer with a uniform decoder and a channel optimized encoder, along with a uniform source on $[0,1]$, and a binary symmetric channel with bit error probability ϵ.

Theorem 1. *An encoder optimized uniform quantizer with the Natural Binary Code index assignment has an effective channel code rate of* $r_c = 1$ *for all* $n \geq 1$ *and* $\epsilon \in [0, 1/2)$.

Theorem 2. *For* $n \geq 2$*, an encoder optimized uniform quantizer with the Modified Natural Code index assignment has an effective channel code rate of*

$$r_c = \begin{cases} 1 & \textit{for} \;\; \epsilon \in [0, \epsilon_n^*) \\ \frac{1}{n}\log\left(2^{n-1}+2\right) & \textit{for} \;\; \epsilon \in [\epsilon_n^*, 1/(2^{n/2}+2)) \\ \frac{n-1}{n} & \textit{for} \;\; \epsilon \in [1/(2^{n/2}+2), 1/2) \end{cases}$$

where $\epsilon_n^* < 2^{-n-1}$ *is the unique root of the polynomial* $-8\epsilon^3 + (4-2^{n+1})\epsilon^2 + (2+2^{n+1})\epsilon - 1$ *in the interval* $(0, 1/2)$.

Theorem 2 shows for any $\epsilon > 0$, for n sufficiently large an encoder optimized uniform quantizer with the Modified Natural Code has half the number of nonempty encoding cells as one with the Natural Binary Code. Theorem 3 shows that despite this fact, the Modified Natural Code and the Natural Binary Code induce the same cell density function.

Theorem 3. *A sequence of encoder optimized uniform quantizers with either the Natural Binary Code or the Modified Natural Code index assignment has a cell density function*

$$\gamma(x) = \begin{cases} \frac{1}{1-2\epsilon} & \textit{for } \epsilon < x < 1-\epsilon \\ 0 & \textit{else.} \end{cases}$$

In [1, 2, 4] it was shown that for all n and all ϵ the Natural Binary Code is optimal for quantizers with no channel optimization and with channel optimized decoders. Theorem 4, however, shows that with a channel optimized encoder, for every $\epsilon > 0$, the Modified Natural Code index assignment outperforms the Natural Binary Code for n sufficiently large. Let $D_{EO}^{(\pi_n)}$ denote the end-to-end MSE of an encoder optimized uniform quantizer with index assignment π_n.

Theorem 4. $D_{EO}^{(MNC)} < D_{EO}^{(NBC)}$ *if and only if* $n \geq 3$ *and* $\epsilon > \hat{\epsilon}_n$*, where* $\hat{\epsilon}_n$ *is the unique root of the polynomial* $4(2^n-4)\epsilon^4 + 2^n(2^n-2)\epsilon^3 - 2(2^{2n}-2^{n+2}-4)\epsilon^2 + 2^n(2^n-4)\epsilon - 1$ *on the interval* $(0,1/2)$*, and* $2^{-2n} < \hat{\epsilon}_n < 2^{-2n+1}$ *when* $n \geq 4$.

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