ECE 45 Discussion 3 Notes

Frequency Response
The inputs and outputs of RLC circuits are generally either voltages or currents. The output of the circuit depends on the frequency of the input.

- An input/output RLC circuit is defined by its *transfer function* (also called frequency response): $H(\omega) = \frac{Out(\omega)}{In(\omega)}$, where $Out(\omega)$ and $In(\omega)$ are the phasor transforms of $out(t)$ and $in(t)$, for an arbitrary frequency $\omega$. The inputs and outputs of RLC circuits are generally either voltages or currents.
- In general, $out(t) \neq in(t) h(t)$ (a common mistake in this course).
- $H(\omega)$ is a complex number, which is a function of $\omega$, since input/output relationships change with different frequencies.
  
  $|H(\omega)| = \sqrt{\text{Re}\{H(\omega)\}^2 + \text{Im}\{H(\omega)\}^2}$ and $\angle H(\omega) = \tan^{-1}\left(\frac{\text{Im}\{H(\omega)\}}{\text{Re}\{H(\omega)\}}\right)$.

- When $x(t)$ is sinusoidal, $x(t) = A \cos(\omega_0 t + \phi)$, and its phasor representation is $Ae^{j\phi}$. If we send it through a system, $H(\omega)$, then the output phasor $Y = A|H(\omega_0)|e^{j(\phi + \angle H(\omega_0))}$, so $y(t) = A|H(\omega_0)| \cos(\omega_0 t + \phi + \angle H(\omega_0))$.

- When $x(t) = \sum A_i \cos(\omega_i + \phi_i)$, by super position, the output is $y(t) = \sum A_i |H(\omega_i)| \cos(\omega_i + \phi_i + \angle H(\omega_i))$.

Example 1:

Find the transfer function $H(\omega)$ of an RLC circuit, with input $x(t)$ and output $y(t)$, given by

$$\frac{d^2y(t)}{dt^2} + x(t) = \frac{dx(t)}{dt} + y(t).$$

Determine the output when the input $x(t) = \cos(2t - \pi/4)$.

Assume $x(t)$ is sinusoidal, then we can represent the differential equation using phasors:

$$(j\omega)^2 Y + X = j\omega X + Y$$

and so

$$H(\omega) = \frac{Y}{X} = \frac{j\omega - 1}{(j\omega)^2 - 1} = \frac{1}{j\omega + 1}.$$
When \( x(t) = \cos(2t) \), the output is given by

\[
y(t) = |H(2)| \cos(2t - \pi/4 + \angle H(2))
\]

where \( |H(2)| = \frac{1}{\sqrt{5}} \) and \( \angle H(2) = \tan^{-1}(2) \).

Example 2:

1. Find the frequency response, \( H(\omega) \), of the circuit below with input \( i_{\text{in}}(t) \) and output \( i_{\text{o}}(t) \).
2. Find the value \( \omega_0 \) for which \( |H(\omega)| \) is maximized.
3. Find the values \( \omega_1 \) such that \( |H(\omega_1)| = |H(\omega_0)|/\sqrt{2} \)
4. With the value of \( R \) fixed, what value should \( L' \) take so that we have \( |H(10^5)| = |H(\omega_0)|/\sqrt{2} \)

![Circuit Diagram]

1. We can use a current divider to find the current through the inductor:

\[
I_o = I_i \frac{\frac{1}{Z_L}}{\frac{1}{Z_R} + \frac{1}{Z_L}} = \frac{1}{\frac{1}{Z_L} + 1}
\]

and so

\[
H(\omega) = \frac{I_o}{I_i} = \frac{1}{1 + j\omega(L/R)}
\]

2. \( |H(\omega)| = \frac{1}{\sqrt{1 + (\omega L/R)^2}} \)
   \( \rightarrow |H|_{\text{max}} = |H(0)| = 1 \)

3. \( |H(\omega_1)| = \frac{1}{\sqrt{1 + (\omega_1 L/R)^2}} = \frac{1}{\sqrt{2}} \rightarrow 1 + (\omega_1 L/R)^2 = 2 \)
   \( \omega_1 = R/L = 10^5/10^{-3} = 10^8 \)

4. \( 10^5 = 10^5/L \rightarrow L = 1H \)
Example 3:

\[ R_1 = 2\Omega, \quad R_2 = 1\Omega, \quad \text{and} \quad C = 1/2F. \]

1. When \( v_2(t) = 0 \) and \( v_1(t) \) is the input to the circuit, find the transfer function \( H_1(\omega) = V_1/V_o \).
2. When \( v_1(t) = 0 \) and \( v_2(t) \) is the input to the circuit, find the transfer function \( H_2(\omega) = V_2/V_o \).
3. Find the output \( v_o(t) \) when \( v_1(t) = 2 \cos(3t + \pi/3) \) and \( v_2(t) = 3 \sin(2t + \pi) \)

Solutions TBD

Example 4:
For an RLC circuit with frequency response

\[ H(\omega) = \begin{cases} 
1 - 2|\omega| & |\omega| \leq 1/2 \\
0 & \text{otherwise}
\end{cases} \]

find the output \( y(t) \) when the input is

\[ x(t) = \sum_{k=0}^{\infty} \frac{1}{1+k} \cos(kt/3). \]

For \( k = 0, 1, 2, \ldots \), let \( x_k(t) = \frac{1}{1+k} \cos(kt/3) \), and let \( y_k(t) \) be the output of the system when \( x_k(t) \) is the input. Then

\[ y_k(t) = |H(k/3)| \frac{1}{1+k} \cos(kt/3 + \angle H(k/3)) \]

and we have \( |H(k/3)| = 0 \) for \( k \geq 2 \), and \( H(0) = 1 \), and \( H(1/3) = 1/3 \), so

\[ y_0(t) = 1 \quad \text{and} \quad y_1(t) = \frac{1}{6} \cos(t/3) \]

By super-position/linearity, we have

\[ y(t) = \sum_{k=0}^{\infty} y_k(t) = 1 + \frac{1}{6} \cos(t/3). \]