Topics

- Complex Numbers
- Functions and Calculus

Why do we care about Complex Numbers?

While values such as voltage and current are always purely real numbers, it is often very useful to model time-dependent voltages and currents using mathematical transformations such as phasors, the Fourier Series, and the Fourier Transform (3 of 4 the main topics in the course).

A strong understanding of complex numbers is critical to using and understanding these math tools that simplify circuit analysis.

- Imaginary Numbers: \( j = \sqrt{-1} \Rightarrow \sqrt{-X} = j \sqrt{X} \)

- Complex Numbers: Consist of real and imaginary parts, which can be represented in either rectangular form or polar form.

- Rectangular Form:
  \[ Z = X + jY \]
  
  \( X \) and \( Y \) are real numbers which are the real and imaginary components of \( Z \).

- Polar Form:
  \[ Z = |Z| e^{j\theta} \]
  
  \( |Z| \) and \( \theta \) are real numbers which are the magnitude and phase components of \( Z \), where \( |Z| \geq 0 \) and \( 0 \leq \theta < 2\pi \).

- Converting Between Representations:
  \[ |Z| = \sqrt{X^2 + Y^2}, \quad \theta = \arctan^{-1}(Y/X), \quad X = |Z| \cos(\theta), \quad Y = |Z| \sin(\theta). \]

These follow directly from trigonometry and viewing \( Z \) as a point on a real and imaginary plot. Plotting a complex number is a useful for checking calculations.

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• Complex Conjugate:

\[ Z = X + jY = |Z| e^{j\theta} \iff Z^* = X - jY = |Z| e^{-j\theta} \]

\[ ZZ^* = (|Z| e^{j\theta}) (|Z| e^{-j\theta}) = |Z|^2 \]

• Adding/Subtracting Complex Numbers:

\[ Z_1 \pm Z_2 = (X_1 \pm X_2) + j(Y_1 \pm Y_2) \]

Adding/subtracting is much easier in rectangular form.

• Multiplying Complex Numbers:

\[ Z_1 Z_2 = (X_1 X_2 - Y_1 Y_2) + j(X_1 Y_2 + X_2 Y_1) = |Z_1||Z_2| e^{j(\theta_1 + \theta_2)} \]

• Dividing Complex Numbers:

\[ \frac{Z_1}{Z_2} = \frac{X_1 + jY_1}{X_2 + jY_2} = \frac{|Z_1| e^{j\theta_1}}{|Z_2| e^{j\theta_2}} = \frac{|Z_1|}{|Z_2|} e^{j(\theta_1 - \theta_2)} \]

Multiplying/dividing is much easier in polar form.

• Euler’s Formula:

\[ e^{j\theta} = \cos(\theta) + j\sin(\theta) \]

Follows directly from rectangular vs. polar representations. Can be used to show:

\[ \cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \quad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \]

which are useful for simplifying expressions and deriving trig identities.
Problems:

1. Represent the following complex numbers in polar form: \( Z_1 = 1 - j \) and \( Z_2 = -1 + j \)

   Solution:
   
   \[
   Z_1 = \sqrt{2} \left( \frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}} \right) = \sqrt{2} e^{-j\pi/4} \\
   Z_2 = \sqrt{2} \left( -\frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}} \right) = \sqrt{2} e^{j3\pi/4}
   \]

   We need to be careful of minus signs, since \(-\frac{1}{1} = \frac{1}{1}\), but \(\theta_1 \neq \theta_2\). We can always plot a complex number in the complex plane to ensure our angles make sense.

2. Represent the following complex number in rectangular form: \( e^{j2\pi/3} \)

   Solution:
   
   \[
   \frac{e^{j2\pi/3}}{1 + j\sqrt{3}} = \frac{e^{j2\pi/3}}{2(1/2 + j\sqrt{3}/2)} = \frac{1}{2} \left( \frac{1}{2} + j \frac{\sqrt{3}}{2} \right) = \frac{1}{4} + j\frac{\sqrt{3}}{4}
   \]

3. Find the magnitude and phase of \( f(\omega) = \frac{1}{(j\omega + 1)^2} \)

   Solution:
   
   \[
   j\omega + 1 = \sqrt{1 + \omega^2} e^{j\tan^{-1}\omega} \implies f(\omega) = \frac{1}{(1 + \omega^2) e^{2j\tan^{-1}\omega}} = \frac{e^{-2j\tan^{-1}\omega}}{1 + \omega^2}
   \]
   
   \[
   \Rightarrow |f(\omega)| = \frac{1}{1 + \omega^2}, \quad \angle f(\omega) = -2 \tan^{-1} \omega
   \]

4. Simplify the following expression for a general integer \( n \): \( (\cos(\pi n) + j \sin(\pi n))^n \)

   Solution:
   
   \[
   \cos(\pi n) = (-1)^n, \quad \text{for all integer } n \\
   \sin(\pi n) = 0, \quad \text{for all integer } n
   \]
   
   \[
   \therefore (\cos(\pi n) + j \sin(\pi n))^n = ((-1)^n)^n = (-1)^{n^2} = (-1)^n.
   \]
5. Find the magnitude and phase of \( f(x) \) where \( f(x) = 1 \) when \( x \geq 0 \) and \( f(x) = -j \) when \( x < 0 \).

Solution:

\[
|f(x)| = 1
\]
\[
\angle f(x) = \begin{cases} 
0 & x \geq 0 \\
-\pi/2 & x < 0
\end{cases}
\]

6. Evaluate the following integral: \( \int_{0}^{\infty} e^{-x} \cos(x) \, dx \).

Solution:
Note that for any real numbers \( a, b > 0 \), we have

\[
\lim_{x \to \infty} e^{-x(a+jb)} = \left( \lim_{x \to \infty} e^{-ax} \right) \left( \lim_{x \to \infty} e^{-jbx} \right) = 0 \left( \lim_{x \to \infty} e^{-jbx} \right) = 0.
\]

Using Euler’s formula we have

\[
\int_{0}^{\infty} e^{-x} \cos(x) \, dx = \int_{0}^{\infty} e^{-x} \frac{e^{jx} + e^{-jx}}{2} \, dx
\]
\[
= \frac{1}{2} \left( \int_{0}^{\infty} e^{-x(1-j)} + e^{-x(1+j)} \, dx \right)
\]
\[
= \frac{1}{2} \left( \frac{e^{-x(1-j)}}{-1+j} + \frac{e^{-x(1+j)}}{-1-j} \right) \bigg|_{0}^{\infty}
\]
\[
= -\frac{1}{2} \left( \frac{e^{-x(1-j)}(1+j) + e^{-x(1+j)}(1-j)}{(1-j)(1+j)} \right) \bigg|_{0}^{\infty}
\]
\[
= \frac{1}{2} \left( 1+j + (1-j) \right) = \frac{1}{2}
\]

Alternatively, we could use integration by parts and solve for \( \int_{0}^{\infty} e^{-x} \cos(x) \, dx \).

7. Find the derivative with respect to \( x \) of \( \frac{1}{j+x} \).

Solution:

\[
\frac{d}{dx} (j + x)^{-1} = -(j + x)^{-2} = -\frac{1}{(x + j)^2}
\]

8. If \( f(x) = 1 \) for \( x \) between \(-3\) and \( 2\) and is zero otherwise, for what values of \( x \) does \( f(4x) = 1 \)?
   For what values of \( x \) does \( f(x/2 - 3) = 1 \)?

Solution:
\( f(4x) = 1 \) when \( (4x) \) is between \(-3\) and \( 2\). Or when \( x \) is between \(-3/4\) and \( 1/2\).
\( f(x/2 - 3) = 1 \) when \( (x/2 - 3) \) is between \(-3\) and \( 2\). Or when \( x \) is between \( 0 \) and \( 10\).
9. Let \( f(x) = \cos(x) \) when \( x \geq 0 \) and \( f(x) = 0 \) when \( x < 0 \). Let \( g(x, y) = 1 \) when \( x < y \) and \( g(x, y) = 0 \) when \( x \geq y \). Evaluate the following integral in terms of \( y \): \( \int_{-\infty}^{\infty} f(x) g(x, y) \, dx \).

**Solution:**
Since \( f(x) = 0 \) when \( x < 0 \) and \( f(x) = \cos(x) \) when \( x \geq 0 \), we have

\[
\int_{-\infty}^{\infty} f(x) g(x, y) \, dx = \int_{-\infty}^{0} 0 \, dx + \int_{0}^{\infty} \cos(x) g(x, y) \, dx = \int_{0}^{\infty} \cos(x) g(x, y) \, dx
\]

\( g(x, y) = 0 \) when \( x \geq y \), so if \( y < 0 \), then \( \int_{0}^{\infty} \cos(x) g(x, y) \, dx = 0 \), since \( x \geq 0 \).
If \( y \geq 0 \), we have

\[
\int_{0}^{\infty} \cos(x) g(x, y) \, dx = \int_{0}^{y} \cos(x) \, dx + \int_{y}^{\infty} 0 \, dx = \sin(y)
\]

Thus

\[
\int_{-\infty}^{\infty} f(x) g(x, y) \, dx = \begin{cases} 
\sin(y) & y \geq 0 \\
0 & y < 0 
\end{cases}
\]

10. For any positive real number \( a \), evaluate \( \int_{0}^{\infty} x e^{-(a+j)x} \, dx \)

**Solution:**
Recall integration by parts:

\[
\int u \, dv = uv - \int v \, du
\]

Let \( u = x \) and \( dv = e^{-(a+j)x} \). Then \( du = dx \) and \( v = -e^{-(a+j)x}/(a+j) \). So

\[
\int_{0}^{\infty} x e^{-(a+j)x} \, dx = \left. -\frac{e^{-(a+j)x}}{a+j} \right|_{0}^{\infty} - \int_{0}^{\infty} -e^{-(a+j)x} \frac{1}{a+j} \, dx
\]

\[
= \frac{1}{a+j} \left. e^{-(a+j)x} \right|_{0}^{\infty} = -e^{-(a+j)x} \frac{1}{(a+j)^2} \left. \right|_{0}^{\infty} = \frac{1}{(a+j)^2}
\]