Problem 3.1

Given a string of four bits representing a binary number (e.g. 0001 = 1, 0010 = 2, ...), where the bits are independent and the $k$th bit (for $k = 0, 1, 2, 3$) is 1 with a probability of $p^k$, for some $0 < p < 1$. Are the $0$th and $1$st bit independent given that the number falls between 7 and 9?

Let $B_i$ be the event the $i$th bit is a 1, and let $E$ be the even the number falls between 7 and 9.

We have $P[B_i] = p^i$

We know that $B_0$ and $B_1$ are independent, i.e. $P[B_0 B_1] = P[B_0] P[B_1]$, but are they independent given $E$? i.e. is $P[B_0 B_1 | E]$ equal to $P[B_0 | E] P[B_1 | E]$?

$$E = \text{“the number is 7, 8, or 9”} = B_3 B_2 B_1 B_0 \cup B_3 B_2 B_1' B_0 \cup B_3' B_2 B_1 B_0$$

which is a disjoint union, so

$$P[E] = P[B_3 B_2 B_1 B_0] + P[B_3 B_2 B_1' B_0] + P[B_3' B_2 B_1 B_0]$$

$B_1, B_2, B_3$ are independent, so


$$= (1 - p^4) (p^3) (p^2) (p) + (p^4) (1 - p^3) (1 - p^2) (1 - p) + (p^4) (1 - p^3) (1 - p^2) (p)$$

$$= (p^6 - p^{10}) + (p^4 - p^5 - p^6 + p^8 + p^9 - p^{10}) + (p^5 - p^7 - p^8 + p^{10})$$

$$= p^4 - p^7 + p^9 - p^{10}.$$ 

Thus we have

$$P[B_0 | E] = \frac{P[B_0 E]}{P[E]} = \frac{P[\text{“the number is 7 or 9”}]}{P[E]} = \frac{(p^6 - p^{10}) + (p^5 - p^7 - p^8 + p^{10})}{p^4 - p^7 + p^9 - p^{10}} = \frac{p + p^2 - p^3 - p^4}{1 - p^3 + p^5 - p^6}$$

$$P[B_1 | E] = \frac{P[B_1 E]}{P[E]} = \frac{P[\text{“the number is 7”}] p^{10}}{p^4 - p^7 + p^9 - p^{10}} = \frac{p^2 - p^6}{1 - p^3 + p^5 - p^6}$$

$$P[B_1 B_0 | E] = \frac{P[B_1 B_0 | E]}{P[E]} = \frac{P[\text{“the number is 7”}]}{P[E]} = \frac{p^2 - p^6}{1 - p^3 + p^5 - p^6}.$$ 

Clearly $P[B_1 B_0 | E] \neq P[B_0 | E] P[B_1 | E]$, so $B_0$ and $B_1$ are not conditionally independent, given $E$. 
Problem 3.2: For \( N > q \), assume \( q \) items are randomly assigned to \( N \) boxes. Each box is equally likely and may have more than one item assigned to it.

1. What is \( K(N, q) \), the probability that all \( q \) items are assigned to unique boxes?
   The first item can be assigned to any box. There are \( N \) total boxes and \( N \) possible choices for item 1. The second item can be assigned to any box, except the box the first item is in. There are \( N \) total boxes and \( N - 1 \) choices for item 2. The third item can be assigned to any box, except the boxes the first and second item are in. There are \( N \) total boxes and \( N - 2 \) choices for item 3. And so on... The \( q \)th item can be assigned to any box, except the boxes the first \( q - 1 \) items are in. There are \( N \) total boxes and \( N - (q - 1) \) choices for item \( q \). Thus
   \[
   K(N, q) = P[\text{all } q \text{ items are in distinct boxes}]
   = \frac{N}{N} \cdot \frac{N - 1}{N} \cdot \frac{N - 2}{N} \cdots \frac{N - (q - 1)}{N} = \frac{N!}{N^q (N - q)!} = \prod_{i=1}^{q-1} \left(1 - \frac{i}{N}\right)
   \]

2. What is the minimum number of people needed to have the probability that at least two people share a birthday be at least 0.5?
   \[
   0.5 \leq P[\text{at least two people share a birthday}] = P[\text{not all birthdays are distinct}]
   = 1 - P[\text{all birthdays are distinct}] = 1 - K(365, q)
   \]
   So \( K(365, q) \leq 0.5 \). Unfortunately, we do not have a simple inverse formula to solve for \( q \). However, guessing and checking different values of \( q \) gives \( K(365, 23) \approx 0.493 \). This implies we only need about 23 people to have the probability that at least two people share a birthday be at least 0.5. Note that \( K(365, 50) \approx 0.03 \), so the probability that in a group of 50 people, at least two people share a birthday is about 0.97.

Problem 3.3: Suppose there are three traffic lights in a row, designed such that there is an 80% chance the driver will encounter the same color light at next light as the previous (i.e. if the first light is green, the second light has 80% chance of being green). Assume the first light is green or red with 50% probability. What are the probabilities of the events

1. A driver is not stopped at any lights?
   Let \( G_1, G_2, G_3 \) denote the events the first, second, and third lights are green respectively.
   \[
   P[G_1G_2G_3] = P[G_1] P[G_2|G_1] P[G_3|G_1, G_2] = \frac{1}{2} \cdot \frac{4}{5} \cdot \frac{4}{5} = \frac{8}{25}
   \]

2. A driver is stopped at two or more lights?
   \[
   P[\text{two or more lights are red}]
   = P[G_1 G_2^c G_3] + P[G_1^c G_2 G_3^c] + P[G_1^c G_2^c G_3] + P[G_1^c G_2 G_3^c]
   = \left(\frac{1}{2} \cdot \frac{1}{5} \cdot \frac{4}{5}\right) + \left(\frac{1}{2} \cdot \frac{4}{5} \cdot \frac{1}{5}\right) + \left(\frac{1}{2} \cdot \frac{1}{5} \cdot \frac{4}{5}\right) + \left(\frac{1}{2} \cdot \frac{4}{5} \cdot \frac{1}{5}\right) = \frac{1}{2}
   \]

3. A driver is stopped at at least one light?
   \[
   P[\text{not all lights are green}] = 1 - P[\text{all lights are green}] = 1 - P[G_1G_2G_3] = 1 - \frac{8}{25} = \frac{17}{25}
   \]
Problem 3.4

Assume each component $R_1, \ldots, R_{10}$ fails with probability $0 < (1 - p) < 1$, independently of any other component in the circuit. We say the circuit works (i.e. does not fail), if there exists a path from $a$ to $b$ such that every component along the path works.

1. Determine the probability the circuit works

Let $E_i$ denote the event $R_i$ works. Let $X$ and $Y$ denote the event that the upper and lower paths, respectively, work. The circuit works if either the upper path or the lower path works. We have

$$X = (E_1 E_2 \cup E_3 E_4) E_5 = (E_1 E_2 E_5) \cup (E_3 E_4 E_5)$$
$$Y = (E_6 E_7 \cup E_8 E_9) E_{10} = (E_6 E_7 E_{10}) \cup (E_8 E_9 E_{10})$$

and so

$$P[X] = P[E_1 E_2 E_5] + P[E_3 E_4 E_5] - P[E_1 E_2 E_3 E_4 E_5]$$
$$= 2p^3 - p^5$$

(Similarly $P[Y] = 2p^3 - p^5$)

Note $X$ and $Y$ are independent, since $P[X|Y] = P[X]$. i.e. whether the upper branch works has no impact on whether the lower branch works. So we have

$$P[\text{Circuit Works}] = P[X \cup Y] = P[X] + P[Y] - P[XY]$$
$$= 4p^3 - 2p^5 - 4p^6 + p^{10}$$

2. Determine the probability the circuit works, assuming $R_5$ fails.

Intuitively, if we assume $R_5$ fails, the probability the circuit works should be the probability the lower branch works.

$$P[X \cup Y | E^c_5] = \frac{P[(X \cup Y) E^c_5]}{P[E^c_5]} = \frac{P[X E^c_5 \cup Y E^c_5]}{1 - p} = \frac{P[Y E^c_5]}{1 - p}$$
$$= \frac{P[Y | E^c_5]}{1 - p} = \frac{P[Y](1 - p)}{1 - p} = P[Y] = 2p^3 - p^5$$

3. Determine the probability $R_5$ fails, assuming the circuit works.

$$P[E^c_5 | X \cup Y] = \frac{P[E^c_5 (X \cup Y)]}{P[X \cup Y]} = \frac{P[X \cup Y | E^c_5]}{P[X \cup Y]}$$
$$= \frac{P[Y] P[E^c_5]}{P[X \cup Y]} = \frac{1 - p}{2 - P[Y]} = \frac{1 - p}{2(1 - p^3) + p^5}$$