INSTRUCTIONS
This exam is closed book and closed notes, with the one exception that you are allowed notes on both sides of one sheet of 8.5 x 11.0 inch paper (not tape or stick-ons allowed). No calculators, laptop computers, or other electronic devices are allowed.

Write your answers in the spaces provided. Show all your work. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress on a problem. Zero credit will be given for correct answers that lack adequate explanation of how they were obtained. There is a maximum total of 55 points on this exam. Simplify your answers as much as possible and leave answers as fractions, not as decimal numbers.
Problem 1 (20 points)
The circuit shown below has one resistor, one inductor, and two current sources (one pointing down and one pointing up). Find the sinusoidal steady-state output voltage $v(t)$ as a function of time $t$ indicated in the diagram. Simplify your answer as much as possible and make sure your answer is purely real-valued. If an expression cannot be simplified by hand, it can be left in a simplified form with square roots, trigonometric functions, etc.
**Problem 1** (continued)

**SOLUTION:**

Consider the more general circuit where the current sources are

\[ i_1(t) = -\cos(\omega_1 t + \phi_1) \]
\[ i_2(t) = \cos(\omega_2 t + \phi_2) \]

and with resistor value \( R \) and inductor value \( L \). Since we have two different frequencies \( \omega_1 \) and \( \omega_2 \), use superposition. First, we zero out (by opening) the current source \( i_2(t) \). Then, we zero out (by opening) the current source \( i_1(t) \). This makes the remaining current source go through the parallel combination of the resistor and inductor capacitor. Thus, we get in phasors:

\[ V_1 = I_1 \cdot (R \parallel L j \omega_1) = -e^{j \phi_1} \cdot \frac{RLj\omega_1}{R + Lj\omega_1} \]
\[ V_2 = I_2 \cdot (R \parallel L j \omega_2) = e^{j \phi_2} \cdot \frac{RLj\omega_2}{R + Lj\omega_2} \]

Thus,

\[ v_1(t) = -\frac{RL\omega_1}{\sqrt{R^2 + (L\omega_1)^2}} \cos(\omega_1 t + \phi_1 + (\pi/2) - \tan^{-1}(L\omega_1/R)) \]
\[ v_2(t) = \frac{RL\omega_2}{\sqrt{R^2 + (L\omega_2)^2}} \cos(\omega_2 t + \phi_2 + (\pi/2) - \tan^{-1}(L\omega_2/R)) \]

So in the special case of: \( \omega_1 = 3, \omega_2 = 4, R = 6, L = 12, \phi_1 = 0, \phi_2 = -\pi/2 \) we get

\[ v_1(t) = -\frac{216}{\sqrt{36 + 36^2}} \cos(3t + (\pi/2) - \tan^{-1} 6) = \frac{36}{\sqrt{37}} \sin(3t - \tan^{-1} 6) \]
\[ v_2(t) = \frac{288}{\sqrt{36 + 48^2}} \cos(4t - (\pi/2) + (\pi/2) - \tan^{-1} 8) = \frac{48}{\sqrt{65}} \cos(4t - \tan^{-1} 8) \]
\[ \therefore v(t) = v_1(t) + v_2(t) \]
\[ = \frac{36}{\sqrt{37}} \sin(3t - \tan^{-1} 6) + \frac{48}{\sqrt{65}} \cos(4t - \tan^{-1} 8) \]
Problem 1 (continued)
Problem 2 (20 points)

The function \( f(t) \) is periodic and even and is shown in the figure below. Find its Fourier Series in both exponential form and in trigonometric form. Simplify as much as possible.

\[ f(t) \]

\[ \cdots \]

\[ B \]

\[ \cdots \]

SOLUTION: Suppose, more generally, that the figure is labeled as shown below.

The midpoint of \( t_2 \) and \( t_3 \) is half the period, so we have \( T/2 = (t_2 + t_3)/2 \), i.e.

\[ T = t_2 + t_3. \]

The period of \( f \) is \( T \), so \( \omega_0 = 2\pi/T \). The Fourier Series for \( f \) is
\[ f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \]

\[ F_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t} dt \]

\[ F_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt \]

\[ = \frac{1}{T} \left[ -A \int_{-t_1}^{t_1} dt + B \int_{-T/2}^{t_2} dt + B \int_{t_2}^{T/2} dt \right] \]

\[ = \frac{1}{T} \left[ -2At_1 + 2B((T/2) - t_2) \right] \]

\[ = B - \frac{2At_1 + t_2}{T} \]

For \( n \neq 0 \):

\[ F_n = \frac{1}{T} \left[ -A \int_{-t_1}^{t_1} e^{-jn\omega_0 t} dt + B \int_{-T/2}^{-t_2} e^{-jn\omega_0 t} dt + B \int_{t_2}^{T/2} e^{-jn\omega_0 t} dt \right] \]

\[ = \frac{1}{T} \left[ -\frac{A}{jn\omega_0} (e^{-jn\omega_0 t_1} - e^{jn\omega_0 t_1}) + \frac{B}{jn\omega_0} (e^{jn\omega_0 t_2} - e^{jn\omega_0 T/2}) + \frac{B}{jn\omega_0} (e^{-jn\omega_0 T/2} - e^{-jn\omega_0 t_2}) \right] \]

\[ = \frac{2}{nT\omega_0} \left[ -A \sin(n\omega_0 t_1) - B \sin(n\omega_0 t_2) + B \sin(n\pi) \right] \]

\[ = \frac{1}{\pi n} \left[ A \sin(n\omega_0 t_1) + B \sin(n\omega_0 t_2) \right] \]

In exponential form:

\[ f(t) = B - \frac{2At_1 + t_2}{T} - \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{A \sin(n\omega_0 t_1) + B \sin(n\omega_0 t_2)}{n} \cdot e^{jn2\pi t/(t_2 + t_3)} \]

In trigonometric form, notice that \( F_n = F_{-n} \) for all \( n \), so

\[ f(t) = F_0 + \sum_{n=1}^{\infty} F_n (e^{jn\omega_0 t} + e^{-jn\omega_0 t}) = F_0 + 2 \sum_{n=1}^{\infty} F_n \cos(n\omega_0 t) \]

\[ = B - \frac{2At_1 + t_2}{T} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{A \sin(n\omega_0 t_1) + B \sin(n\omega_0 t_2)}{n} \cdot \cos(2\pi nt/(t_2 + t_3)) \]
Problem 2 (continued)
Problem 2 (continued)
Problem 3 (15 points)

(If you get the wrong answer for any of parts (a)-(c) of this problem you get zero points for that part. If you get the correct answer, you only get non-zero points for good justification.)

(a) (5 points) True or False. (circle one)
If \( x(t) \) and \( y(t) \) are each real-valued periodic functions with period \( T \), then \( x(t) + y(t) \) is also periodic with the same period \( T \).

SOLUTION:
False. Let \( x(t) = \sin t + \sin(2t) \) and \( y(t) = -\sin t + \sin(2t) \). Then \( x(t) \) and \( y(t) \) each have period \( 2\pi \) but \( x(t) + y(t) = 2\sin(2t) \) which has period \( \pi \).

(b) (5 points) True or False. (circle one)
A system that produces an output signal \( x(\cos(t^2)) \) for an arbitrary input signal \( x(t) \) is a linear system.

SOLUTION:
True. The output for an input \( x_1(t) \) is \( y_1(t) = x_1(\cos(t^2)) \) and the output for an input \( x_2(t) \) is \( y_2(t) = x_2(\cos(t^2)) \). The output for an input \( x(t) = Ax_1(t) + Bx_2(t) \) is \( y(t) = x(\cos(t^2)) = Ax_1(\cos(t^2)) + Bx_2(\cos(t^2)) = Ay_1(t) + By_2(t) \).
Problem 3 (continued)

(c) (5 points) True or False. (circle one)
If a linear time-invariant system with frequency response \( H(\omega) \) has a periodic input signal \( x(t) \) with exactly two non-zero Fourier Series coefficients, then the output of the system has a Fourier Series with exactly two non-zero Fourier Series coefficients.

SOLUTION:
False. Suppose the input is \( x(t) = \cos(t) \). Its Fourier Series is \((1/2)e^{jt} + (1/2)e^{-jt}\) which has exactly two non-zero coefficients, and has period \( T = 2\pi \) and \( \omega_0 = 1 \). Suppose the LTI system is a high-pass filter with frequency response
\[
H(\omega) = \begin{cases} 
1 & \text{if } |\omega| > 10 \\
0 & \text{if } |\omega| \leq 10
\end{cases}
\]

Then the output signal is
\[
y(t) = \sum_{n=-\infty}^{\infty} X_n H(n\omega_0) e^{jn\omega_0 t} = (1/2)e^{jt} H(1) + (1/2)e^{-jt} H(-1) = 0.
\]

which does not have exactly two non-zero Fourier Series coefficients.