Fourier Series Properties

We can represent a periodic function as a sum of sinusoidal components with different coefficients, i.e. a Fourier Series. However, the calculations of the coefficients is often tedious and time-consuming. We can use the properties of the Fourier series to simplify calculations of similar signals. Proving each of these properties is a good exercise.

Fourier Series as an Input to an LTI System

A Fourier series is a sum of sinusoidal components. We know how to analyze LTI systems for sinusoidal inputs, so by linearity, we can determine the output when we have a periodic input. Let \( x(t) \) be periodic with fundamental frequency \( \omega_0 \), then

\[
x(t) = \sum_{n=-\infty}^{\infty} X_n e^{j\omega_0 nt} \rightarrow H(\omega) \rightarrow y(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) X_n e^{j\omega_0 nt}
\]

Notice that \( y(t) \) is also a periodic function with fundamental frequency \( \omega_0 \). We can represent \( y(t) \) using a Fourier series with coefficients:

\[
Y_n = X_n H(n\omega_0)
\]

Linearity and Scaling

If we have two periodic functions \( f(t) \) and \( g(t) \) with fundamental frequency \( \omega_0 \) and coefficients \( F_n \) and \( G_n \), respectively.

Let \( h(t) = a f(t) + b g(t) \)

\( h(t) \) will also be periodic with fundamental frequency \( \omega_0 \) and coefficients:

\[
H_n = a F_n + b G_n
\]

Time Reversal

If we have two periodic functions \( f(t) \) and \( g(t) \) such that \( g(t) = f(-t) \), then \( G_n = F_{-n} \).

Time Shifting

If we have two periodic functions \( f(t) \) and \( g(t) \) such that \( g(t) = f(t - t_0) \), then \( G_n = F_n e^{-j\omega_0 t_0} \).

Time Scaling

If we have two periodic functions \( f(t) \) and \( g(t) \) such that \( g(t) = f(at) \) and the fundamental frequency of \( f(t) \) is \( \omega_0 \), then \( G_n = F_n \), but the fundamental frequency of \( g(t) \) is \( a\omega_0 \).
Time Derivative

If we have two periodic functions $f(t)$ and $g(t)$ such that $g(t) = \frac{df(t)}{dt}$, then $G_n = j\omega_0 n F_n$.

Time Multiplication

If we have three periodic functions $f(t)$, $g(t)$, and $h(t)$ such that $h(t) = f(t) g(t)$, the coefficients for $h(t)$ can be determined from the coefficients for $f(t)$ and $g(t)$

$$H_n = \sum_{k=-\infty}^{\infty} F_k G_{n-k}$$

Parseval’s Theorem

The average power in a period of a periodic function can be calculated two ways:

$$P_{avg} = \frac{1}{T} \int_T |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |F_n|^2$$

Sometimes the calculations for one method are much simpler than the other.

Real Functions

If $f(t)$ is a real function, then $F_n = F^*_{-n}$. The Fourier series of a real function can be simplified to a sum of sines and cosines.
This follows from the fact that for a real function $f(t) = f^*(t)$.

Imaginary Functions

If $f(t)$ is an imaginary function, then $F_n = -F^*_{-n}$
This follows from the fact that for a real function $f(t) = -f^*(t)$.

Even Functions

If $f(t)$ is an even function, then $F_n = F_{-n}$. The Fourier series of an even function can be simplified to a sum of cosines.
This follows from the fact that for an even function $f(t) = f(-t)$.

Odd Functions

If $f(t)$ is an odd function, then $F_n = -F_{-n}$. The Fourier series of an odd function can be simplified to a sum of sines.
This follows from the fact that for an even function $f(t) = -f(-t)$. 
Examples

1. Write $f(t)$ as a sum of sines and cosines and find the average power in a period, where

$$f(t) = \begin{cases} A & \text{for } -1 < t < 1 \\ \vdots & \end{cases}$$

We note that $f(t)$ has period 2 and in the period $[-1, 1)$, $f(t) = At$. So for all $n \neq 0$, we have

$$F_n = \frac{1}{T} \int_T f(t) e^{-j\omega_0 nt} dt = \frac{A}{2} \int_{-1}^{1} te^{-j\pi nt} dt$$

$$= \frac{A}{-j2\pi n} \left( te^{-j\pi nt} \bigg|_{-1}^{1} - \int_{-1}^{1} e^{-j\pi nt} dt \right)$$

$$= \frac{-A}{j2\pi n} \left( e^{-j\pi n} + e^{j\pi n} + \frac{1}{j\pi n} \left( e^{-j\pi n} - e^{j\pi n} \right) \right) = \frac{-A}{j\pi n} (-1)^n.$$ 

In the second line, we divide by 0 when $n = 0$, so

$$F_0 = \frac{A}{2} \int_{-1}^{1} t e^0 dt = 0.$$ 

Thus we have

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j\omega_0 nt} = \sum_{n=1}^{\infty} \frac{-A}{j\pi n} (-1)^n e^{j\pi n} - \frac{-A}{j\pi n} (-1)^{-n} e^{-j\pi n}$$

$$= -A \sum_{n=1}^{\infty} (-1)^n \left( \frac{e^{j\pi n} - e^{-j\pi n}}{j\pi n} \right) = -2A \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi n} \sin(\pi nt)$$

The average power in a period is a straight-forward calculation in the time domain:

$$\frac{1}{2} \int_{-1}^{1} f(t)^2 dt = \frac{A^2}{2} \int_{-1}^{1} t^2 dt = \frac{A^2}{3}.$$ 

By Parseval’s Theorem:

$$\frac{A^2}{3} = \sum_{n=-\infty}^{\infty} |F_n|^2 = \sum_{n=-\infty}^{\infty} \left| \frac{-A}{j\pi n} (-1)^n \right|^2 = \sum_{n=-\infty}^{\infty} \frac{A^2}{\pi^2 n^2} = \sum_{n=1}^{\infty} \frac{2A^2}{\pi^2 n^2}$$

Thus we have

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is a beautiful bit of mathematics :)
2. Write \( g(t) \) as a sum of sines and cosines and find the Fourier series components \( G_n \), where

\[
1 - A \leq g(t) \leq A + 1
\]

Note that we have \( g(t) = -f(2t - 2) + 1 \), so

\[
g(t) = 1 + 2A \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi n} \sin(\pi n (2t - 2)) = 1 + A \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi n} \sin(2\pi nt).
\]

In order to find \( G_n \) in terms of \( F_n \), let’s use some intermediate steps:

Let \( x(t) = -f(t - 1) \), then \( x(t) \) is periodic with period 2 and \( X_n = -F_n e^{-j\pi n} \).

Let \( y(t) = x(2t) \), then \( y(t) \) is periodic with period 1 and \( Y_n = X_n \).

Then \( g(t) = y(t) + 1 \), so for \( n \neq 0 \),

\[
G_n = -F_n (-1)^n = \frac{A}{j\pi n}
\]

and \( G_0 = F_0 + 1 = 1 \).

3. Suppose \( g(t) \) is the input to an LTI system with frequency response \( H(\omega) = 1 \), when \( |\omega| > \pi \) and is 0 otherwise. Plot the output \( w(t) \).

Since \( g(t) \) is periodic with period 1, \( w(t) \) is also periodic with period 1 and the Fourier series coefficients of \( w(t) \) are given by

\[
W_n = H(\omega_0 n)G_n = H(2\pi n)G_n = \begin{cases} G_n & \text{if } |2\pi n| > \pi \\ 0 & \text{otherwise} \end{cases}
\]

So we have \( W_n = G_n \) for all \( n \neq 0 \) and \( W_0 = 0 \), which implies \( w(t) = g(t) - G_0 = g(t) - 1 \).
4. Calculate the Fourier series components $X_n$ of $x(t)$. Simplify the expression for $X_n$ to be purely real.

$x(t)$ is a periodic function with period $T = 4\pi$ and fundamental frequency $\omega_0 = 1/2$, and

$$x(t) = \begin{cases} 
  \cos t & |t| \leq \pi/2 \\
  0 & \pi/2 < t < 7\pi/2
\end{cases}$$

So for all $n \neq \pm 2$, we have

$$TX_n = \int_{-\pi/2}^{\pi/2} x(t)e^{-j\omega_0 nt} dt = \int_{-\pi/2}^{\pi/2} \cos(t)e^{-jnt/2} dt$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} e^{j(1-n/2)t} + e^{-j(1+n/2)t} dt$$

$$= \frac{1}{2} \left( \frac{e^{j(1-n/2)t}}{j(1-n/2)} + \frac{e^{-j(1+n/2)t}}{-j(1+n/2)} \right)|_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2j} \left( \frac{e^{j\pi/2}e^{-j\pi n/4} - e^{-j\pi/2}e^{j\pi n/4}}{1-n/2} - \frac{e^{-j\pi/2}e^{j\pi n/4} - e^{j\pi/2}e^{j\pi/4}}{1+n/2} \right)$$

$$= \frac{1}{2j} \left( \frac{je^{-j\pi n/4} + je^{j\pi n/4}}{1-n/2} + \frac{je^{j\pi n/4} + je^{-j\pi n/4}}{1+n/2} \right)$$

$$= \cos(\pi n/4) \left( \frac{1}{1-n/2} + \frac{1}{1+n/2} \right)$$

$$= \cos(\pi n/4) \frac{2}{4-n^2}$$

Since we divide by zero when $n = \pm 2$, we must calculate $F_2$ and $F_-2$ separately, but since $x(t)$ is even, so $X_n$ is also even, so $X_2 = X_{-2}$.

$$TX_2 = \frac{1}{2} \int_{-\pi/2}^{\pi/2} 1 + e^{-2jt} dt$$

$$= \left[ \frac{\pi}{2} + \frac{e^{-2jt}}{-2j} \right]|_{-\pi/2}^{\pi/2}$$

$$= \frac{\pi}{2} + \frac{e^{j\pi} - e^{-j\pi}}{-2j} = \pi/2.$$ 

Thus for all integers $n$, we have

$$X_n = \begin{cases} 
  1/4 & n = \pm 2 \\
  \frac{\cos(\pi n/4)}{2\pi(4-n^2)} & n \neq \pm 2
\end{cases}$$
5. Express the Fourier series components of \( z(t) \) in terms of the Fourier series components of \( x(t) \).

\( z(t) \) is a periodic function with period \( 4\pi \). Note that
\[
z(t) = 1 - x(t + \pi/2).
\]
Hence for all \( n \neq 0 \), we have
\[
Z_n = -e^{jn\pi/4}X_n
\]
and \( Z_0 = 1 - X_0 \).

6. Express the Fourier series components of \( r(t) \) in terms of the Fourier series components of \( x(t) \).

What is the DC component of \( r(t) \)? (i.e. \( R_0 \))

\( r(t) \) is a periodic function with period \( 4\pi \). Note that taking the derivative of \( x(t) \) yields
\[
\frac{d}{dt}x(t)
\]
Then multiplying by \(-1\) and shifting this function to the left by \( \pi \) gives us \( r(t) \). Hence
\[
r(t) = -\frac{d}{dt}x(t - \pi).
\]
Thus by the linearity, time-shifting, and time derivative properties, for all integers \( n \), we have
\[
R_n = -\left(\frac{j^n}{4}\right)e^{-jn\pi/4}X_n
\]
By plugging \( n = 0 \) into the expression for \( R_n \), we get \( R_0 = 0 \). Alternatively, recall that
\[
R_0 = \frac{1}{T} \int_T r(t) \, dt
\]
which is the average value of \( r(t) \) in a period \( T \). The average value of \( r(t) \) is clearly 0.