ECE 45 Discussion 3 Notes

Frequency Response

The inputs and outputs of RLC circuits are generally either voltages or currents. The output of the circuit depends on the frequency of the input.

- An input/output RLC circuit is defined by its frequency response (also called the transfer function): \( H(\omega) = \frac{Out(\omega)}{In(\omega)} \), where \( Out(\omega) \) and \( In(\omega) \) are the phasor transforms of \( out(t) \) and \( in(t) \), for an arbitrary frequency \( \omega \). The inputs and outputs of RLC circuits are generally either voltages or currents.

- In general, \( out(t) \neq in(t) \times H(\omega) \) (a common mistake in this course).

- \( H(\omega) \) is a complex number, which is a function of \( \omega \), since input/output relationships change with different frequencies.

\[
|H(\omega)| = \sqrt{\text{Re}\{H(\omega)\}^2 + \text{Im}\{H(\omega)\}^2} \quad \text{and} \quad \angle H(\omega) = \tan^{-1}\left(\frac{\text{Im}\{H(\omega)\}}{\text{Re}\{H(\omega)\}}\right).
\]

- Suppose \( x(t) \) is sinusoidal, \( x(t) = A \cos(\omega_0 t + \phi) \), so its phasor representation is \( Ae^{j\phi} \). If \( x(t) \) is the input to an RLC circuit with frequency response \( H(\omega) \), then the output phasor is \( Y = A |H(\omega_0)| e^{j(\phi + \angle H(\omega_0))} \), so

\[
y(t) = A |H(\omega_0)| \cos(\omega_0 t + \phi + \angle H(\omega_0)).
\]

The frequency response describes how the system responds to a sinusoid of a particular frequency.

- When \( x(t) = \sum A_i \cos(\omega_i + \phi_i) \), by super position, the output is

\[
y(t) = \sum A_i |H(\omega_i)| \cos(\omega_i + \phi_i + \angle H(\omega_i)).
\]

- This allows us to analyze RLC circuits whose inputs are voltages and currents consisting of multiple frequencies.

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Example 1

Find the frequency \( H(\omega) \) of an RLC circuit, with input \( x(t) \) and output \( y(t) \), given by

\[
\frac{d^2y(t)}{dt^2} + x(t) = \frac{dx(t)}{dt} + y(t).
\]

Determine the output when the input \( x(t) = \cos(2t - \pi/4) \).

Solutions

Assume \( x(t) \) is sinusoidal, then we can represent the differential equation using phasors:

\[
(j\omega)^2 Y + X = j\omega X + Y
\]

and so

\[
H(\omega) = \frac{Y}{X} = \frac{j\omega - 1}{(j\omega)^2 - 1} = \frac{1}{j\omega + 1}.
\]

When \( x(t) = \cos(2t) \), the output is given by

\[
y(t) = |H(2)| \cos(2t - \pi/4 + \angle H(2))
\]

where \( |H(2)| = \frac{1}{\sqrt{5}} \) and \( \angle H(2) = \tan^{-1}(2) \).

Example 2

(a) Find the frequency response, \( H(\omega) \), of the circuit below with input \( i_{in}(t) \) and output \( i_o(t) \).

(b) Find the value \( \omega_0 \) for which \( |H(\omega)| \) is maximized.

(c) Find the value \( \omega_1 > 0 \) such that \( |H(\omega_1)| = |H(\omega_0)|/\sqrt{2} \)

(d) With the value of \( R \) fixed, what value should \( L \) take so that we have \( |H(10^4)| = |H(\omega_0)|/\sqrt{2} \)

\[ \begin{align*}
R & = 100 \text{ k}\Omega, \quad L = 1 \text{ mH} \\
\end{align*} \]

Solutions

(a) We can use a current divider to find the current through the inductor:

\[
I_o = I_i \frac{1}{\frac{Z_L}{Z_R} + \frac{1}{Z_L}} = \frac{1}{\frac{Z_L}{Z_R} + 1} \quad \text{and so} \quad H(\omega) = \frac{I_o}{I_i} = \frac{1}{1 + j\omega(L/R)}
\]
(b) To maximize $|H(\omega)|$ with respect to $\omega$, note that

$$|H(\omega)| = \frac{1}{\sqrt{1 + (\omega L/R)^2}} \rightarrow |H|_{\text{max}} = |H(0)| = 1$$

(c) $|H(\omega_1)| = \frac{1}{\sqrt{1 + (\omega_1 L/R)^2}} = \frac{1}{\sqrt{2}} \rightarrow 1 + (\omega_1 L/R)^2 = 2 \rightarrow \omega_1 = R/L = 10^5/10^{-3} = 10^8 \times 10^4 = \omega_1 = R/L = 10^5/L \rightarrow L = 10H$

Example 3

\[ \begin{array}{c}
R_1 & \text{v}_1(t) & + \\
\text{C} & \text{v}_o(t) & \text{v}_2(t)
\end{array} \]

$R_1 = 2\Omega$, $R_2 = 1\Omega$, and $C = 1/2F$.

(a) When $v_2(t) = 0$ and $v_1(t)$ is the input to the circuit, find the frequency response $H_1(\omega) = V_o/V_1$.

(b) When $v_1(t) = 0$ and $v_2(t)$ is the input to the circuit, find the frequency response $H_2(\omega) = V_o/V_2$.

(c) Find the output $v_o(t)$ when $v_1(t) = 2 \cos(3t + \pi/3)$ and $v_2(t) = 3 \sin(2t)$

Solutions

(a) When $v_2(t) = 0$, by using a voltage divider on the phasor-transformed circuit, we have

$$V_o = V_1 \frac{(R_2/\left(1/j\omega C\right))}{R_1 + (R_2/\left(1/j\omega C\right))} = V_1 \frac{R_2}{R_1 + R_2 + j\omega CR_1R_2}$$

and so

$$H_1(\omega) = \frac{V_o}{V_1} = \frac{1}{3 + j\omega} = \frac{1}{\sqrt{9 + \omega^2}} e^{-j\tan^{-1}(\omega/3)}$$

(b) Similarly, when $v_1(t) = 0$, by using a voltage divider on the phasor-transformed circuit, we have

$$V_o = V_2 \frac{(R_1/\left(1/j\omega C\right))}{R_2 + (R_1/\left(1/j\omega C\right))} = V_2 \frac{R_1}{R_1 + R_2 + j\omega CR_1R_2}$$

and so

$$H_2(\omega) = \frac{V_o}{V_2} = \frac{2}{3 + j\omega} = \frac{2}{\sqrt{9 + \omega^2}} e^{-j\tan^{-1}(\omega/3)}$$
(c) We know that when $v_2(t) = 0$ and $v_1(t) = 2\cos(3t + \pi/3)$, the output is

$$v_{o,1}(t) = 2|H_1(3)|\cos(3t + \pi/3 + \angle H_1(3))$$

and when $v_1(t) = 0$ and $v_2(t) = 3\sin(2t)$, the output is

$$v_{o,2}(t) = 3|H_2(2)|\sin(2t + \angle H_2(2)).$$

So by super-position, the output is

$$v_o(t) = v_{o,1}(t) + v_{o,2}(t) = \frac{2}{\sqrt{18}}\cos(3t + \pi/3 - \tan^{-1}(1) + \frac{6}{\sqrt{13}}\sin(2t - \tan^{-1}(2/3))$$

$$= \frac{\sqrt{2}}{3}\cos(3t + \pi/12) + \frac{6}{\sqrt{13}}\sin(2t - \tan^{-1}(2/3))$$

Example 4

Suppose the output of an RLC circuit is

$$\frac{1}{\sqrt{16 + \omega^2}}\cos(\omega t - \tan^{-1}(\omega/3))$$

when the input is $\cos(\omega t)$, where $\omega \geq 0$.

(a) What is the output when the input is $A\cos(\omega t + \theta)$ for some real numbers $A, \theta$?

(b) What is the output when the input is $2 + \sin(4t)$?

Solutions

(a) When the input to an RLC circuit is $\cos(\omega t)$, the output is

$$|H(\omega)|\cos(\omega t + \angle H(\omega)).$$

Hence

$$H(\omega) = \frac{e^{-j\tan^{-1}(\omega/4)}}{\sqrt{16 + \omega^2}}$$

and the output when $A\cos(\omega t + \theta)$ is

$$\frac{A}{\sqrt{16 + \omega^2}}\cos(\omega t + \theta - \tan^{-1}(\omega/4)).$$

This is a useful trick to use in general. That is, we can find the output when $\cos(\omega t)$ is the input, then apply any scaling and shifting after the fact. This can simplify some of the phasor analysis of circuits. This utilizes the linearity and time invariance of an RLC circuit.
(b) Let \( x_1(t) = 2 \) and \( x_2(t) = \sin(4t) \), and suppose \( y_1(t) \) and \( y_2(t) \) are the outputs when \( x_1(t) \) and \( x_2(t) \), respectively, are the inputs. Then by superposition, when \( x_1(t) + x_2(t) \) is the input, \( y_1(t) + y_2(t) \) is the output.

\( x_1(t) \) is the case in (a), where \( A = 2, \omega = 0, \) and \( \theta = 0 \), so \( y_1(t) = \frac{2}{4} \).

\( x_2(t) \) is the case in (b), where \( A = 1, \omega = 4, \) and \( \theta = -\pi/2 \), so
\[
y_2(t) = \frac{1}{\sqrt{32}} \cos(4t - \pi/2 - \tan^{-1}(1)).
\]

Thus the desired output is
\[
\frac{1}{2} + \frac{1}{4\sqrt{2}} \sin(4t - \pi/4)
\]

**Example 5**

*For an RLC circuit with frequency response*

\[
H(\omega) = \begin{cases} 
1 - 2j\omega & |\omega| \leq 1/2 \\
0 & \text{otherwise}
\end{cases}
\]

*find the output \( y(t) \) when the input is*

\[
x(t) = \sum_{k=0}^{\infty} \frac{1}{1+k} \cos(kt/3).
\]

**Solutions**

For \( k = 0, 1, 2, \ldots \), let \( x_k(t) = \frac{1}{1+k} \cos(kt/3) \), and let \( y_k(t) \) be the output of the system when \( x_k(t) \) is the input. Then, since \( H(\omega) \) is the frequency response of an RLC circuit,

\[
y_k(t) = |H(k/3)| \frac{1}{1+k} \cos(kt/3 + \angle H(k/3))
\]

and we have \( |H(k/3)| = 0 \) for \( k \geq 2 \), and \( H(0) = 1 \), and \( H(1/3) = 1 - 2j/3 = \sqrt{1 + 4/9} e^{j\tan^{-1}(-2/3)} \), so

\[
y_0(t) = 1 \quad \text{and} \quad y_1(t) = \frac{\sqrt{7}}{6} \cos(t/3 - \tan^{-1}(2/3))
\]

By super-position/linearity, we have

\[
y(t) = \sum_{k=0}^{\infty} y_k(t) = 1 + \frac{\sqrt{7}}{6} \cos(t/3 - \tan^{-1}(2/3)).
\]