Problem 3.1:
(a) Let \( X \) be a continuous random variable with probability density function:

\[
f(u) = \begin{cases} 
    K_i & \text{if } a_i < u < b_i \ (1 \leq i \leq m) \\
    0 & \text{else}
\end{cases}
\]

where \( b_i \leq a_{i+1} \) for \( i = 1, \ldots, m - 1 \). Suppose we have an \( N \)-point scalar quantizer for \( X \), which consists of an \( N_i \)-point uniform quantizer in the interval \((a_i, b_i)\) for each \( i = 1, \ldots, m \), where \( N_1 + \ldots + N_m = N \). For each \( i \), determine the fraction \( N_i/N \) of codepoints allocated to the interval \((a_i, b_i)\), such that the overall mean-squared error \( E[(X - Q(X))^2] \) is minimized. Make as few assumptions as necessary and state them explicitly.

(b) Consider the special case when \( m = 2, a_1 = 0, b_1 = 1, a_2 = 5, b_2 = 7, K_1 = 1/2, K_2 = 1/4 \). For each \( N = 2, \ldots, 1000 \), perform an exhaustive computer search over all possible choices of \( N_1 \) and \( N_2 \) to determine the optimal fraction \( N_i/N \). Plot the resulting \( N_i/N \) as a function of \( N \) and indicate on the graph the value of \( N_i/N \) predicted by your answer in part (a).

Problem 3.2:
Prove that the unique MSE-optimal \( N \)-point quantizer for a uniform source on \((a, b)\) is a uniform quantizer on \((a, b)\) with cell widths \( \frac{b-a}{N} \).

Problem 3.3:
Suppose \( a < b < c \), and let \( X \) be a discrete random variable taking on the values \( a, b, \) and \( c \), with probabilities \( I, J \), and \( K \), respectively. Let \( y_1 \) and \( y_2 \) be the codepoints of a MSE-optimal 2-point scalar quantizer for \( X \). Assume \( y_1 < y_2 \). Determine the values of \( y_1 \) and \( y_2 \) as functions of \( a, b, c \) and \( I, J, K \). Simplify your answer as much as possible.

Problem 3.4:
Suppose \( a < b < c < d \), and let \( X \) be a continuous random variable with probability density function

\[
f(u) = \begin{cases} 
    I & \text{if } a < u < b \\
    J & \text{if } c < u < d \\
    0 & \text{else}
\end{cases}
\]

Determine the MSE-optimal 2-point scalar quantizer as a function of \( a, b, c, d, I, J \).

Problem 3.5:
Let \( X \) be a continuous random variable with finite mean and variance and with probability density function \( f \) such that \( f(u) = 0 \) for all \( u < 0 \) and \( f(u) > 0 \) for all \( u > 0 \). Suppose we quantize \( X \) using an \( N \)-point uniform quantizer on the interval \([0, B]\). Any values of \( X \) larger than \( B \) get quantized by the right-most codepoint of the uniform quantizer. Let \( B \) be chosen to minimize the MSE \( e \). You may use high resolution approximations for the uniform quantizer in \([0, B]\).

(a) Prove that \( B \to \infty, B/N \to 0 \), and \( e \to 0 \) as \( N \to \infty \).
(b) Determine the asymptotic growth rate that $B \to \infty$ as a function of $N$, as $N \to \infty$ for the following source densities:

$$f_1(u) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-u^2/2} & \text{if } u > 0 \\ 0 & \text{else} \end{cases}$$

$$f_2(u) = \begin{cases} e^{-u} & \text{if } u > 0 \\ 0 & \text{else} \end{cases}$$

If you cannot get an exact rate, use reasonable approximations to get an accurate estimate.