Conditional Probability

- The **conditional probability** of event $E$ given event $F$ is

$$P(E|F) = \frac{P(EF)}{P(F)}$$

This definition is only valid when $P(F) > 0$.

- **Fact:** $P(EF) = P(E|F)P(F) = P(F|E)P(E)$

- A collection of events in sample space $S$ is said to **cover** $S$ if their union equals $S$. A collection of events $A_1, A_2, \ldots, A_n$ is called a **partition** of $S$ if the sets are disjoint and cover $S$.

- If the events $A_1, A_2, \ldots, A_n$ partition $S$, then

$$E = EA_1 \cup EA_2 \cup \cdots \cup EA_n$$

$$P(E) = P(EA_1) + \cdots + P(EA_n)$$

$$= P(E|A_1)P(A_1) + \cdots + P(E|A_n)P(A_n)$$

$$P(A_i|E) = \frac{P(E|A_i)P(A_i)}{\sum_{k=1}^{n} P(E|A_k)P(A_k)}$$

- Similarly, for conditional probability, we have

$$P(E|F) = P(E|A_1F)P(A_1|F) + \cdots + P(E|A_nF)P(A_n|F).$$

- Conditional probability satisfies:
  - **Axiom 1:** $0 \leq P(E|F) \leq 1$.
  - **Axiom 2:** $P(S|F) = 1$.
  - **Axiom 3:** If $E_1, E_2, E_3, \ldots$ are disjoint, then

$$P(E_1 \cup E_2 \cup E_3 \cup \ldots |F) = P(E_1|F) + P(E_2|F) + P(E_3|F) + \ldots.$$  

- **Chain Rule:**

$$P(E_1 E_2 E_3 \ldots E_n) = P(E_1)P(E_2|E_1)P(E_3|E_2E_1)\ldots P(E_n|E_1E_2\ldots E_{n-1})$$

- There are many different ways of asking for $P(E|F)$ in words. The following statements are all equivalent:
  - “Find the probability of $E$ given $F$”
  - “What is the probability $E$ occurred, if $F$ occurred?”
  - “If $F$ occurred, then determine the probability that $E$ occurred.”
  - “You learn that $F$ happened. Now what is the probability of $E$?”