ECE 109: Solutions to Problem Set #7

1. Let $X$ and $Y$ be independent random variables, uniform over $[0, 1]$, and let $Z = X - Y$. Then

$$f_Z(w) = \int_{-\infty}^{\infty} f_X(u) f_Y(u-w) \, du = \int_{0}^{1} f_Y(u-w) \, du.$$ 

Therefore, $f_Z(w) = 0$ for $|w| \geq 1$; $f_Z(w) = 1+w$ for $-1 \leq w < 0$; $f_Z(w) = 1-w$ for $0 \leq w < 1$.

2. We compute $F_{X,Y}(u,v)$ for positive $u,v$. For $v \leq u$, $\{Y \leq v\}$ implies $\{X \leq u\}$. Therefore, for $v \leq u$, $F_{X,Y}(u,v) = \Pr(X \leq u, Y \leq v) = \Pr(Y \leq v) = \Pr(W \leq v, Z \leq v) = (1-e^{-v})^2$, the last equality because $W$ and $Z$ are independent. For $v > u$,

$$\{X \leq u, Y \leq v\} = \{W \leq u, Z \leq v\} \cup \{W \leq v, Z \leq u\} \cup \{u < W \leq v, Z \leq u\},$$

where the last three events are disjoint. Therefore,

$$F_{X,Y}(u,v) = \Pr(X \leq u, Y \leq v) = (1-e^{-u})^2 + 2(1-e^{-u})(e^{-u}-e^{-v}).$$

Taking derivatives, we obtain $f_{X,Y}(u,v) = 2e^{-(u+v)}$ for $0 \leq u < v < \infty$; $f_{X,Y}(u,v) = 0$ otherwise.

3. The joint pdf $f_{X,Y}(a,b)$ is uniform on the square $|a| + |b| \leq 1$, whose diagonal has length 2, so the square’s side length is $\sqrt{2}$ and its area is 2. This implies the pdf has value $1/2$ inside the square. Let $W = X + Y$ and $Z = X - Y$. Now compute the joint CDF of $Z$ and $W$. First suppose $u,v \in [-1,1]$. Then,

$$F_{W,Z}(u,v) = \Pr[W \leq u, Z \leq v] = \Pr[X+Y \leq u, X-Y \leq v] = \Pr(\{(a,b) : a+b \leq u, \ a-b \leq v\}).$$

This last probability is the area of a rectangle (rotated 45 degrees), times the pdf 1/2.
One side of the rectangle is the line segment connecting the point \((-1, 0)\) to the intersection of the lines \(a + b = -1\) and \(a - b = v\) (colored green). This intersection occurs at \(\left(\frac{v-1}{2}, \frac{(v+1)}{2}\right)\), and the length of this side of the rectangle is \(\frac{v+1}{\sqrt{2}}\).

The other side of the rectangle is the line segment connecting \((-1, 0)\) to the intersection of the lines \(a - b = -1\) and \(a + b = u\) (colored red). This intersection occurs at \(\left(\frac{u-1}{2}, \frac{(u+1)}{2}\right)\), and the length of this side of the rectangle is \(\frac{u+1}{\sqrt{2}}\).

Thus, the area of the rectangle is \(\frac{u+1}{\sqrt{2}} \cdot \frac{v+1}{\sqrt{2}}\) and therefore the CDF is

\[
F_{W,Z}(u, v) = \frac{(u + 1)(v + 1)}{4}.
\]

The pdf can be computed as follows:

\[
\frac{\partial F_{W,Z}(u, v)}{\partial u} = \frac{v + 1}{4},
\]

\[
f_{W,Z}(u, v) = \frac{\partial^2 F_{W,Z}(u, v)}{\partial v \partial u} = \frac{\partial}{\partial v} \left( \frac{v + 1}{4} \right) = \frac{1}{4}
\]

All of this so far assumed \(u, v \in [-1, 1]\).

Now suppose \(u \in [-1, 1]\) and \(v > 1\). Then, \(F_{W,Z}(u, v) = \frac{u+1}{2}\) which by taking two derivatives implies \(f_{W,Z}(u, v) = 0\). If, on the other hand, \(u \in [-1, 1]\) and \(v < -1\), then \(F_{W,Z}(u, v) = 0\) which also implies \(f_{W,Z}(u, v) = 0\). Similarly for the cases when \(v \in [-1, 1]\) and \(|u| > 1\).

In summary the joint pdf of \(W\) and \(Z\) is

\[
f_{W,Z}(u, v) = \begin{cases} 
1/4 & \text{if } |u|, |v| \leq 1 \\
0 & \text{else.}
\end{cases}
\]

4. Let \(f_{X,Y}(u, v) = 2\) if \((u, v) \in [0, 1/2] \times [0, 1/2]\) or \((u, v) \in [1/2, 1] \times [1/2, 1]\) and let \(f_{X,Y}(u, v) = 0\) elsewhere.

5.

\[
1/c = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(u, v) \, du \, dv = \int_{0}^{\infty} \int_{-v}^{v} (v^2 - u^2)e^{-v} \, du \, dv = \frac{4}{3} \int_{0}^{\infty} v^3 e^{-v} \, dv = 8,
\]

where the last integral can be done by parts. Therefore, \(c = 1/8\), and we see also that

\[
f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u, v) \, du = \frac{1}{8} \int_{-v}^{v} (v^2 - u^2)e^{-v} \, du = (1/6)v^3e^{-v}.
\]

Finally, because the density is zero for \(v < |u|\),

\[
f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u, v) \, dv = \frac{1}{8} \int_{|u|}^{\infty} (v^2 - u^2)e^{-v} \, dv = (1/4)e^{-|u|}(1 + |u|),
\]

where, again, the integral can be done by parts.

\[
E[X] = \int_{-\infty}^{\infty} uf_X(u) \, du = \int_{-\infty}^{\infty} u (1/4) e^{-|u|} (1 + |u|) \, du = 0.
\]

Note \(f_X(u)\) is an even function, so this result is expected.
6.  (c)  

\[ \Pr(X > Y) = \frac{6}{7} \int_0^1 \int_0^u \left( u^2 + \frac{uv}{2} \right) \, dv \, du = \frac{15}{56}. \]

(d)  

\[ \Pr(Y > 1/2 | X < 1/2) = \frac{\Pr(Y > 1/2, X < 1/2)}{\Pr(X < 1/2)} = \frac{\left[ \int_{1/2}^1 \int_0^{1/2} (u^2 + uv/2) \, dv \, du \right]}{\left[ \int_0^{1/2} (2u^2 + u) \, du \right]} = 69/80. \]