ECE 109: Solutions to Problem Set #5

1. (a) If A writes 1, then B gains 1 if B writes 1; otherwise B gains -3/4. Thus, B’s expected gain when A writes 1 is \(1p - (3/4)(1-p)\).

(b) If A writes 2, then B gains -3/4 if B writes 1; otherwise B gains 2. Thus, B’s expected gain when A writes 2 is \(-(3/4)p + 2(1-p)\).

The maximin value of the expected gain is attained when the expected gains are equal (plot them as a function of \(p\) if you don’t see why). This occurs at \(p = 11/18\), and the maximin value is 23/72.

Parts (c) and (d) follow from a very similar argument. The minimax loss occurs at \(q = 11/18\).

2. Consider how the event \(\{X = i\}\) can happen. For \(i = 0\), all \(k\) of the oysters that are opened contain pearls; this happens with probability \(p^k\). For \(i = 1\), one of the oysters we open does not contain a pearl; the one without the pearl could be any of the first \(k\) oysters, but not the \((k+1)st\) oyster (make sure you understand why), and each of the \(\binom{k}{i}\) possibilities has probability \(p^k(1-p)\). For \(i = 2\), any two of the first \(k+1\) oysters can fail to have pearls, and each of the \(\binom{k+1}{2}\) possibilities has probability \(p^k(1-p)^2\). More generally,

\[
P(X = r) = \binom{k + r - 1}{r} p^k(1-p)^r.
\]

This probability mass function is directly related to the negative binomial mass function (see Ross, 5th Edition, Section 4.9.2). To show this, we define the random variable \(Y\) to be the total number of oysters opened (with and without pearls) until \(k\) pearls are found. Then \(Y = X + k\) and

\[
P(Y = n) = P(X = n - k) = \binom{n - 1}{n - k} p^k(1-p)^{n-k} = \binom{n - 1}{k - 1} p^k(1-p)^{n-k}.
\]

(a) To show that \(\sum P(X = r) = 1\), note that, with \(q = 1 - p\),

\[
\sum_{r=0}^{\infty} P(X = r) = (1-q)^k \sum_{r=0}^{\infty} \frac{(k+r-1)(k+r-2) \cdots (r+1)}{(k-1)!} q^r
\]

\[
= \frac{(1-q)^k}{(k-1)!} \sum_{r=0}^{\infty} \frac{q^{k-1} d^{k-1} q^{k-1+r}}{d q^{k-1} q^{k-1+r}}
\]

\[
= \frac{(1-q)^k}{(k-1)!} \left[ \frac{q^{k-1} \sum_{r=0}^{\infty} q^r}{1-q} \right]
\]

\[
= \frac{(1-q)^k}{(k-1)!} \left[ \frac{q^{k-1}}{1-q} \right]
\]

\[
= 1.
\]

(b) The mean and variance of a negative binomial r.v. is derived in Ross, Section 4.9.2. Using these results, it is easy to derive that \(E[X] = k[(1/p) - 1]\), \(\text{Var}[X] = k(1-p)/p^2\).
\( P(X = r) = \frac{(k + r - 1)(k + r - 2) \cdots k}{r!} \left(1 - \frac{\lambda}{k}\right)^k \left(\frac{\lambda}{k}\right)^r \)

Letting \( k \to \infty \), we obtain

\[
\lim_{k \to \infty} P(X = r) = \frac{\lambda^r}{r!} e^{-\lambda},
\]

which we recognize as the Poisson distribution.

3. The geometric RV \( X \) has a pmf \( p_X(k) = (1/2)^k \). \( X \) takes on values \( k = 1, 2, 3, \ldots \) and therefore, \( Y = \sin(\pi X/2) \) takes on values \( \{-1, 0, 1\} \) for \( k = 4m + 3, 2m \), and \( 4m + 1 \) respectively. Therefore, \( Y \) is a discrete RV and its pmf can be computed as follows.

\[
\begin{align*}
p_Y(1) &= \frac{1}{2^1} + \frac{1}{2^5} + \frac{1}{2^9} + \cdots = \frac{1/2^1}{1 - (1/2^4)} = \frac{8}{15} \\
p_Y(-1) &= \frac{1}{2^3} + \frac{1}{2^7} + \frac{1}{2^{11}} + \cdots = \frac{1/2^3}{1 - (1/2^4)} = \frac{2}{15} \\
p_Y(0) &= 1 - p_Y(1) - p_Y(-1) = \frac{1}{3}
\end{align*}
\]

Thus the pdf of \( Y \) is given by

\[
p_Y(k) = \begin{cases} 
\frac{2}{15}, & k = -1 \\
\frac{1}{3}, & k = 0 \\
\frac{8}{15}, & k = 1
\end{cases}
\]

4. In this problem, \( c = 3 \).

(a) \( P(6X^2 - 5X - 1 > 0) = P((6X + 1)(X - 1) > 0) = P(\{X < -1/6\} \cup \{X > 1\}) = 0 \)

since \( f_X(u) = 0 \) for \( u /\in (0, 1) \).

\( P(6X^2 - 7X + 2 > 0) = P((3X - 2)(2X - 1) > 0) = P(\{X < 1/2\} \cup \{X > 2/3\}) = 1 - P(1/2 \leq X \leq 2/3) = 1 - \int_{1/2}^{2/3} 3(1-u)^2 \, du = 197/216. \)

(b) \( E[X] = \int_0^1 u^3(1-u)^2 \, du = 1/4. \ E[X^2] = \int_0^1 u^2(1-u)^2 \, du = 1/10. \) Hence, \( \text{Var}[X] = E[X^2] - (E[X])^2 = 1/10 - 1/16 = 3/80. \)

5. Let \( X \) be the width of a screw. Then \( X \sim \mathcal{N}(0.9, 9.0 \times 10^{-6}). \)

(a) This is asking for \( P(\{X < 0.895\} \cup \{X > 0.905\}) = 2F_X(0.895) \) (why?), and \( 2F_X(0.895) = 2\Phi(-1.667) = 2(1 - \Phi(1.667)) = 0.095 \), or 9.5\%. The maximum allowable value of \( \sigma \) is the maximum \( \sigma \) that satisfies \( 2[1 - \Phi(0.005/\sigma)] \leq 0.01. \) That is, \( \sigma \leq 0.0019. \)

(b) This is asking for \( P(\{X < 0.896\} \cup \{X > 0.908\}) = 1 - [F_X(0.908) - F_X(0.896)] = 1 - [\Phi(2.667) - \Phi(-1.333)] = 2 - \Phi(2.667) - \Phi(1.333) \approx 0.0956 \), or 9.56\%.