1. A and B play the following game: A writes down either number 1 or number 2, and B must guess which one. If the number that A has written down is i and B has guessed correctly, then B receives i units from A. If B makes a wrong guess, then B pays $\frac{3}{4}$ unit to A. If B randomizes his decision by guessing 1 with a probability $p$ and 2 with a probability $1 - p$, determine his expected gain if

(a) A has written down 1?
(b) A has written down 2?

What value of $p$ maximizes the minimum possible value of B’s expected gain, and what is this maximin value? (Note that B’s expected gain depends not only on $p$ but also on what A does).

Consider now player A. Suppose that she also randomizes her decision, writing down number 1 with probability $q$. What is A’s expected loss if

(a) B chooses number 1?
(b) B chooses number 2?

What value of $q$ minimizes A’s maximum expected loss? Show that the minimum of A’s maximum expected loss is equal to the maximum of B’s minimum expected gain.

This result is known as the minimax theorem, and was first established in generality by mathematician John von Neumann. It is the fundamental result in the mathematical discipline known as the theory of games. The common value is called the value of the game to player B.

2. Any oyster contains a pearl with probability $p$ independently of its neighbors and no oyster contains more than one pearl. You open a sequence of oysters until you find exactly $k$ pearls. Let $X$ be total the number of oysters that you opened that did not contain a pearl.

(a) Find $\Pr(X = r)$ and show that $\sum_r \Pr(X = r) = 1$.
(b) Find the mean and variance of $X$.
(c) If $p = 1 - (\lambda/k)$, find the limit of the distribution of $X$ as $k \to \infty$.

3. $X$ is a geometric random variable with parameter $1/2$, and $Y = \sin(\pi X/2)$. Is $Y$ discrete, continuous, or mixed? Find the CDF and either the pdf or pmf of $Y$.

4. The random variable $X$ has probability density function $f_X(u) = c(1 - u)^2$ when $0 < u < 1$, and 0 elsewhere.

(a) Find $\Pr(6X^2 > 5X + 1)$. Find $\Pr(6X^2 > 7X - 2)$.
(b) Find the mean and variance of $X$. 
5. (NOTE: This is one of the few problems in this course where you actually have to work out a decimal answer). The width of a screw produced by a steel company is Gaussian distributed with mean $m = 0.9$ cm and standard deviation $\sigma = 0.0030$ cm.

(a) If the specification limits of a customer are $0.9 \pm 0.0050$ cm, what percentage will be defective? What is the maximum allowable value of $\sigma$ that permits no more than 1 in 100 defectives when the widths are Gaussian distributed with $m = 0.9$ and $\sigma$?

(b) If the specification limits of a customer are a minimum of 0.896 cm and a maximum 0.908 cm, what percentage will be defective?