1. The following table is quite helpful in solving this problem:

<table>
<thead>
<tr>
<th>ω in...</th>
<th>ABC</th>
<th>ABCc</th>
<th>ABc</th>
<th>ABcCc</th>
<th>A&quot;BC</th>
<th>A&quot;BCc</th>
<th>A&quot;BcC</th>
<th>A&quot;BcCc</th>
</tr>
</thead>
<tbody>
<tr>
<td>value of $X(ω)$</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>probability</td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
<td>0.125</td>
</tr>
</tbody>
</table>

(a) Referring to the table above, the random variable $X$ takes on the values $-1, 0, 1, 2, 3$.

(b) Using the table again, we see that the probability mass function of $X$ is given by $p_X(-1) = p_X(3) = 0.125$, $p_X(0) = p_X(1) = p_X(2) = 0.25$, and $p_X(u) = 0$ for all $u$ not in the set $\{-1, 0, 1, 2, 3\}$. The CDF of $X$ can be easily found from its pmf, as follows

$$F_X(u) = \begin{cases} 
0 & u < -1 \\
0.125 & -1 \leq u < 0 \\
0.375 & 0 \leq u < 1 \\
0.625 & 1 \leq u < 2 \\
0.875 & 2 \leq u < 3 \\
1 & u \geq 3 
\end{cases}$$

2. One possible probability measure on $S = \{1, 2, 3, 4, 5\}$ that would satisfy the requirements of parts (a) and (b) is defined by

$$P(\{1\}) = 0.1; \quad P(\{2\}) = 0.1; \quad P(\{3\}) = 0.2; \quad P(\{4\}) = 0.2; \quad P(\{5\}) = 0.4;$$

The solution in what follows is with respect to this probability measure.

(a) We can now define the random variable $X$ as follows

$$X(i) = i \quad \text{for } i = 1, 2, 3, 4, 5 \quad (1)$$

It is immediate from the definition that $X$ takes the values $1, 2, 3, 4, 5$ with probabilities $0.1, 0.1, 0.2, 0.2, 0.4$, as desired. Notice that there are other ways to define the random variable $X$ to satisfy this property. For instance, we could define

$$X(1) = 2; \quad X(2) = 1; \quad X(i) = i \quad \text{for } i = 3, 4, 5 \quad (2)$$

(b) The random variable $Y$ could be defined as follows

$$Y(3) = \sqrt{2}; \quad Y(1) = Y(4) = \sqrt{3}; \quad Y(2) = Y(5) = \pi \quad (3)$$

Again, this definition of $Y$ satisfies the required property, but it is not unique. Why?
(c) It is easy to see that the random variable $Z = XY$ takes 5 different values, depending on the five possible outcomes in $S$. For $X$ and $Y$ defined in (1) and (3), the possible values of $Z$ are
\[ \sqrt{3}, 2\pi, 3\sqrt{2}, 4\sqrt{3}, 5\pi \]
taken with probabilities $0.1, 0.1, 0.2, 0.2, 0.4$ respectively. If the random variables $X, Y$ were defined differently, then the answer could be different. For instance, if $X$ is defined as in (2) then $Z$ takes the values $2\sqrt{3}, \pi, 3\sqrt{2}, 4\sqrt{3}, 5\pi$.

3. (a) Here $F_X(u)$ is not nondecreasing ($F_X(1) = 1; F_X(2) = 0$), and hence not a valid CDF.
(b) Here $F_X(u)$ is a valid CDF. It is continuous everywhere except at $u = 0$, where it is right-continuous. We have $P\{|X| > 0.5\} = P\{X < -0.5\} + P\{X > 0.5\}$ and
\[ P\{X < -0.5\} + P\{X > 0.5\} = F_X(-0.5^+) + [1 - F_X(0.5)] = 0.5e^{-1} + 0.25e^{-1.5} \]
(c) Here $F_X(u)$ is not right-continuous at $u = 0$, and therefore not a valid CDF.

4. (a) By the properties of a pdf we have
\[ 1 = \int_{-\infty}^{+\infty} f_X(u) \, du = c \int_0^3 u \, du + c \int_3^6 (6-u) \, du = 9c \]
Hence $c = 1/9$.
(b) We have a continuous random variable, and therefore $F_X(a) = \int_{-\infty}^a f_X(u) \, du$. Thus
\[ \begin{align*}
a < 0 : & \quad F_X(a) = 0 \\
0 \leq a < 3 : & \quad F_X(a) = \frac{1}{9} \int_0^a u \, du = \frac{a^2}{18} \\
3 \leq a < 6 : & \quad F_X(a) = \frac{1}{9} \int_0^3 u \, du + \frac{1}{9} \int_3^a (6-u) \, du = \frac{1}{2} - \frac{1}{18} (6-u)^2 \bigg|_3^a = 1 - \frac{(6-a)^2}{18} \\
3 \leq a = 6 : & \quad F_X(a) = 1 \\
\end{align*} \]
This function $F_X(u)$ is sketched below.
(c) It is easy to see that $F_X(u)$ is nondecreasing and that $0 \leq F_X(u) \leq 1$ for all $u$, either from the sketch above or by analyzing the functions $u^2/18$ and $1 - (6-u)^2/18$ in the intervals $[0,3]$ and $[3,6]$. Further, in our case $F_X(-\infty) = F_X(0) = 0$ and $F_X(+\infty) = F_X(6) = 1$. Finally, we see that $F_X(u)$ is continuous everywhere, including the points $u = 0, u = 3,$ and $u = 6$.
(d) We have $P(A) = P(X > 3) = 1 - F_X(3) = 0.5$. Similarly $P(B) = P(1.5 \leq X \leq 9) = P(X \leq 9) - P(X < 1.5) = F_X(9) - F_X(1.5^-) = 1 - 0.125 = 0.875$.
(e) The intersection of $A$ and $B$ is the event $\{3 < X \leq 9\}$ and $P(AB) = P(3 < X \leq 9) = P(X > 3) = 0.5 = P(A) \neq P(A)P(B)$. Hence, the two events are not independent.

5. Let $X$ denote the chosen number. Then $X$ is a continuous RV, uniformly distributed on the interval $(0, 1)$. We have
Figure 1:

(a) $P(0.1 \leq X < 0.2) = 0.2 - 0.1 = 0.1$.

(b) $P(\text{second digit } = 2) = \sum_{k=0}^{9} P(0.k2 \leq X < 0.k3) = 10 \cdot 0.01 = 0.1$.

(c) $P(0.3 \leq \sqrt{X} < 0.4) = P(0.09 \leq X < 0.16) = 0.07$.

6. (a) $P(X > 5) = \frac{1}{5} \int_{5}^{\infty} e^{-u/5} \, du = -e^{-u/5} \bigg|_{5}^{\infty} = e^{-1}$.

(b) $P(X < 6) = \frac{1}{5} \int_{0}^{6} e^{-u/5} \, du = -e^{-u/5} \bigg|_{0}^{6} = 1 - e^{-6/5}$.

(c) $P(5 \leq X \leq 6) = \frac{1}{5} \int_{5}^{6} e^{-u/5} \, du = -e^{-u/5} \bigg|_{5}^{6} = e^{-1} - e^{-6/5}$.

(d) $P(X < 6 \mid X > 5) = P(5 < X < 6)/P(X > 5) = (e^{-1} - e^{-6/5})/e^{-1} = 1 - e^{-1/5}$.

Note that this is also the unconditional probability that the conversation takes less than one minute.