ECE 109: Solutions to Problem Set #3

1. It is better to be player B. Here’s why. If player A chooses a, then player B chooses c. If player A chooses b, then player B chooses a. If player A chooses c, then player B chooses b. The probability that a defeats c is $P[a = 9] + P[a = 5, c = 2] = \frac{1}{3} + \frac{1}{2} = \frac{5}{6} < \frac{1}{2}$. The probability that b defeats a is $P[a = 1, b = 8] + P[a = 5, c = 8] + P[a = 1, b = 4] + P[a = 1, b = 3] = 4 \cdot \frac{1}{3} = \frac{4}{3} < \frac{1}{2}$. The probability that c defeats b is $P[b = 3, c = 7] + P[b = 4, c = 7] + P[b = 3, c = 6] + P[b = 4, c = 6] = 4 \cdot \frac{1}{3} = \frac{4}{3} < \frac{1}{2}$. This player B can always guarantee a greater than 1/2 probability of winning.

2. Let $S_i$ be the event that the $i$th card chosen is a spade. Then we want

$$P(S_1|S_2S_3) = \frac{P(S_2S_3|S_1)P(S_1)}{P(S_2S_3)} \quad = \frac{P(S_2S_3|S_1)P(S_1)}{P(S_2S_3|S_1)P(S_1) + P(S_2S_3|S_1^c)P(S_1^c)} \quad = \frac{\left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right)}{\left(\frac{13}{52}\right)\left(\frac{12}{51}\right)\left(\frac{11}{50}\right) + \left(\frac{12}{52}\right)\left(\frac{11}{51}\right)\left(\frac{10}{50}\right)} \quad = \frac{11}{50}$$

3. Let $M$ be the event that the person is male and let $C$ be the event that he or she is color blind. Also, let $p$ denote the proportion of the population that is male. Then

$$P(M|C) = \frac{P(C|M)P(M)}{P(C|M)P(M) + P(C|M^c)P(M^c)} \quad = \frac{(0.05)p}{(0.05)p + (0.0025)(1 - p)}$$

4. Let $W$ and $F$ be the events that component 1 works and that the system functions respectively. Then

$$P(W|F) = \frac{P(F|W)P(W)}{P(F)} = \frac{1 \cdot (1/2)}{1 - P(F^c)} = \frac{1/2}{1 - (1/2)^n}$$

5. Let $B$ be the event that $E$ occurs before $F$. Condition on the outcome of the initial trial. That is, consider the probability of $B$ given that $E$ happened on the first trial or give that $F$ happened on the first trial, or neither happened on the first trial:

$$P(B) = P(B|E)P(E) + P(B|F)P(F) + P(B|E^cF^c)P(E^cF^c) \quad = 1 \cdot P(E) + 0 \cdot P(F) + P(B)(1 - P(E) - P(F)) \quad = P(E) + P(B)(1 - P(E) - P(F))$$
Solving for \( P(B) \) gives \( P(B) = \frac{P(E)}{P(E) + P(F)} \).

6. We have \( P(A) = P(B) = P(C) = 1/2 \) and \( P(AB) = P(AC) = P(BC) = 1/4 \). But \( P(ABC) = 1/4 \).

7. False. Consider a sample space \( S = \{HH, TT, HT, TH\} \) consisting of two independent fair coin tosses. Define events \( A = \{HH, HT\} \), \( B = \{TT, HT\} \), and \( F = \{HH, TT, HT\} \). Then \( P(A|B) = P(AB)/P(B) = (1/4)/(2/4) = 1/2 = P(A) \) and hence \( A \) and \( B \) are independent. Since \( B \subset F \) we have \( BF = B \), and hence \( P(A|BF) = P(A|B) = 1/2 \). But \( P(A|F) = P(AF)/P(F) = P(A)/P(F) = (2/4)/(3/4) = 2/3 \). Hence \( P(A|BF) \neq P(A|F) \) and thus \( A \) and \( B \) are not conditionally independent.