Problem 1 (20 points)

Suppose the probability is \( \frac{T_2 - T_1}{T_1 T_2} \) that the temperature outside is between \( T_1 \) and \( T_2 \), whenever \( 1 \leq T_1 \leq T_2 \).

(a) (5 points) Find the probability density function of the temperature outside.

SOLUTION: Let \( X \) be the temperature outside.
Note that \( P[1 \leq X \leq u] \to 1 \) as \( u \to \infty \), so \( P[X < 1] = 0 \).

\[
F_X(u) = P[X \leq u] = P[1 \leq X \leq u] = 1 - \frac{1}{u} \quad \text{for } u \geq 1
\]

\[
f_X(u) = \frac{d}{du} F_X(u) = \begin{cases} 
  u^{-2} & u \geq 1 \\
  0 & \text{else}
\end{cases}
\]
(b) (5 points) Find the variance of the fourth root of the temperature outside.

**SOLUTION:** Let $X$ be the temperature outside and let $Y = X^{1/n}$.

$$
E[Y] = E[X^{1/n}] = \int_1^{\infty} u^{1/n} u^{-2} du = \int_1^{\infty} u^{(1/n)-2} du = \frac{1}{1 - (1/n)} = \frac{n}{n - 1}
$$

$$
E[Y^2] = E[X^{2/n}] = \int_1^{\infty} u^{2/n} u^{-2} du = \int_1^{\infty} u^{(2/n)-2} du = \frac{1}{1 - (2/n)} = \frac{n}{n - 2}
$$

Var$(Y) = E[Y^2] - (E[Y])^2 = \frac{n}{n - 2} - \left( \frac{n}{n - 1} \right)^2 = \frac{n}{(n - 2)(n - 1)^2}$

If $n = 4$, then Var$(Y) = 2/9$. 
(c) (5 points) Let $X$ be the temperature outside. Find the probability density function of $e^{2X}$.

**SOLUTION:** Let $Y = e^{aX}$. If $u \geq e^a$, then

$$F_Y(u) = P(Y \leq u) = P(e^{aX} \leq u) = P(X \leq \ln(u)/a) = \int_1^{\ln(u)/a} f_X(z)dz$$

$$f_Y(u) = \frac{d}{du} \int_1^{\ln(u)/a} z^{-2}dz = \left(\frac{\ln u}{a}\right)^{-2} \frac{1}{ua} = \frac{a}{u \ln^2 u}$$

$$\therefore f_Y(u) = \begin{cases} \frac{2}{u \ln^2 u} & \text{if } u \geq e^2 \\ 0 & \text{else} \end{cases}$$
(d) (5 points) Find the probability the temperature outside is less than 2, given the temperature outside is less than 4.

**SOLUTION:** Let $a < b$.

$$P(X < a | X < b) = \frac{P(X < a, X < b)}{P(X < b)} = \frac{P(X < a)}{1 - (1/b)} = \frac{b(a - 1)}{a(b - 1)}.$$

Thus, $P(X < 2 | X < 4) = 2/3$. 
Problem 2 (10 points)

A box contains 5 coins of Type A and 2 coins of Type B. All Type A coins are biased with the probability of Heads equal to $1/6$ and all Type B coins are biased with the probability of Heads equal to $1/3$. You pull out two coins from the box at random without replacement. You flip each coin once.

What is the probability that both coins were Type A, given that two Heads were observed?

SOLUTION:

Let there be $a$ coins of Type A and $b$ coins of Type B. Type A coins have probability of Heads $r$ and all Type B coins have probability of Heads $s$. Let $S$ be the event that both coins were Type A. Let $T$ be the event that both coins were Type B. Let $U$ be the event that one coin was Type A and one coin was Type B. Let $V$ be the event that two Heads were observed.

\[
P(V|S) = r^2.
\]

\[
P(V|T) = s^2.
\]

\[
P(V|U) = rs.
\]

\[
P(S) = \frac{a}{a+b} \cdot \frac{a-1}{a+b-1}
\]

\[
P(T) = \frac{b}{a+b} \cdot \frac{b-1}{a+b-1}
\]

\[
P(U) = 2 \cdot \frac{a}{a+b} \cdot \frac{b}{a+b-1}
\]

\[
P(V) = P(V|S)P(S) + P(V|T)P(T) + P(V|U)P(U)
\]

\[
= \frac{a(a-1)r^2}{(a+b)(a+b-1)} + \frac{b(b-1)s^2}{(a+b)(a+b-1)} + \frac{2abr}{(a+b)(a+b-1)}
\]

\[
= \frac{(ar+bs)^2 - ar^2 - bs^2}{(a+b)(a+b-1)}
\]

\[
P(S|V) = \frac{P(V|S)P(S)}{P(V)}
\]

\[
= r^2 \cdot \frac{a(a-1)}{(a+b)(a+b-1)} \cdot \frac{(a+b)(a+b-1)}{(ar+bs)^2 - ar^2 - bs^2}
\]

\[
= \frac{r^2a(a-1)}{(ar+bs)^2 - ar^2 - bs^2} = \frac{a(a-1)}{(a+b(s/r))^2 - a - b(s/r)^2}
\]

If $a = 5, b = 2, s = 1/3, r = 1/6$, then $P(S|V) = \frac{5(5-1)}{[5+2(2)]^2 - 5 - 2(2)^2} = \frac{20}{81-5-8} = \frac{20}{68} = \frac{5}{17}$
Problem 2 (continued)
Problem 3 (10 points)

Let $A$ be the origin in the plane and let $B$ be a point chosen uniformly at random on the circle $x^2 + y^2 = 25$. Let $C$ be the point where a vertical line through $B$ intersects the $x$-axis. What is the expected value of the area of triangle $\triangle ABC$?

SOLUTION:

Point $B$ is chosen such that the random angle $\theta$ that line $AB$ makes (clockwise) with the positive $x$-axis is uniform on $[0, 2\pi]$. Let $r$ be the radius of the circle. Then we have:

$$B = (r \cos \theta, r \sin \theta)$$
$$C = (r \cos \theta, 0)$$

$$\text{Area}(\triangle ABC) = |r \cos \theta| \cdot |r \sin \theta|/2$$
$$= (r^2/2)|\sin \theta \cos \theta|$$
$$= (r^2/4)|\sin(2\theta)|$$

$$f_\theta(u) = 1/(2\pi) \text{ on } [0, 2\pi]$$

$$E[\text{Area}(\triangle ABC)] = \frac{r^2}{4} \int_0^{2\pi} |\sin(2u)| \frac{1}{2\pi} du$$

$$= \frac{r^2}{8\pi} \left( \int_0^{\pi/2} |\sin(2u)| du + \int_{\pi/2}^{\pi} |\sin(2u)| du + \int_{\pi}^{3\pi/2} |\sin(2u)| du + \int_{3\pi/2}^{2\pi} |\sin(2u)| du \right)$$

$$= \frac{r^2}{2\pi} \int_0^{\pi/2} \sin(2u) du$$

$$= \frac{r^2}{2\pi} \cdot \cos(2u) \bigg|_{0}^{\pi/2}$$

$$= \frac{r^2}{2\pi}$$

Note that each of the integrals, on the line above containing 4 integrals, is the same since it is the area under a positive sine for one-half a period.
Problem 3 (continued)