INSTRUCTIONS
This exam is closed book and closed notes, with the one exception that you are allowed notes on both sides of one sheet of 8.5 x 11.0 inch paper (not tape or stick-ons allowed). No calculators, laptop computers, or other electronic devices are allowed.

Write your answers in the spaces provided. Show all your work. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress on a problem. Zero credit will be given for correct answers that lack adequate explanation of how they were obtained. Simplify your answers as much as possible and leave answers as fractions, not as decimal numbers.

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Name ________________________________
Your UCSD ID Number __________________
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Problem 1 (20 points)

Akiko, Bao, and Carla take an exam. The probability Bao passes is 0.57, the probability Carla passes is 0.65, and the probability that both of them pass is 0.27. The probability that Carla passes, but neither Akiko nor Bao pass is 0.08. The probability that Akiko passes, but neither Bao nor Carla pass is 4 times as large as the probability that none of the three of them pass.

(a) (10 points) Find the probability that at least one of Akiko, Bao, or Carla passes.

SOLUTION:

\[ P(B) = 0.57 \]
\[ P(C) = 0.65 \]
\[ P(BC) = 0.27 \]
\[ P(B \cup C) = P(B) + P(C) - P(BC) = 0.57 + 0.65 - 0.27 = 0.95 \]
\[ P(A^c B^c C) = 0.08 \]
\[ P(AB^c C^c) = 4P(A^c B^c C^c) \]
\[ 1 = P(S) = P(B \cup C) + P(AB^c C^c) + P(A^c B^c C^c) \]
\[ = 0.95 + 4P(A^c B^c C^c) + P(A^c B^c C^c) \]
\[ \therefore P(A^c B^c C^c) = 0.05/5 = 0.01 \]
\[ P(A \cup B \cup C) = 1 - P(A^c B^c C^c) = 1 - 0.01 = 0.99 \]
(b) (10 points) Find the probability that either Akiko or Bao (or both) passed, given that either Bao or Carla (or both) passed.

SOLUTION:

\[
P(A \cup B | B \cup C) = \frac{P((A \cup B)(B \cup C))}{P(B \cup C)} = \frac{P(B \cup C) - P(A^cB^cC)}{P(B \cup C)} = \frac{0.95 - 0.08}{0.95} = \frac{87}{95}
\]
Problem 2 (10 points)

A box contains 3 coins of Type A and 2 coins of Type B and a second box contains 5 coins of Type A and 3 coins of Type B. All Type A coins are biased with the probability of Heads equal to \(\frac{3}{4}\) and all Type B coins are biased with the probability of Heads equal to \(\frac{2}{3}\). You randomly pull one coin from each box. You flip each coin once.

What is the probability that both coins were Type A, given that two Tails were observed?

SOLUTION:

Let the first box have \(a\) coins of Type A and \(b\) coins of Type B, and let the second box have \(c\) coins of Type A and \(d\) coins of Type B. Type A coins have probability of Tails \(r\) and all Type B coins have probability of Tails \(s\). Let \(S\) be the event that both coins were Type A.

Let \(T\) be the event that both coins were Type B.

Let \(U\) be the event that one coin was Type A and the other coin was Type B.

Let \(V\) be the event that two Tails were observed.

\[
P(V) = P(V|S)P(S) + P(V|T)P(T) + P(V|U)P(U) \\
= r^2 \cdot \frac{a}{a+b} \cdot \frac{c}{c+d} + s^2 \cdot \frac{b}{a+b} \cdot \frac{d}{c+d} \\
+ rs \cdot \left( \frac{a}{a+b} \cdot \frac{d}{c+d} + \frac{b}{a+b} \cdot \frac{c}{c+d} \right) \\
= \frac{acr^2 + bds^2 + (ad + bc)rs}{(a+b)(c+d)}
\]

\[
P(S|V) = \frac{P(V|S)P(S)}{P(V)} \\
= \frac{r^2 \cdot \frac{a}{a+b} \cdot \frac{c}{c+d}}{P(V)} \\
= \frac{r^2 ac}{r^2 ac + s^2 bd + rs(ad + bc)}
\]

If \(a = 3, b = 2, c = 5, d = 3, s = \frac{1}{3}, r = \frac{1}{4}\), then

\[
P(S|V) = \frac{3 \cdot 5(1/4)^2}{3 \cdot 5(1/4)^2 + 2 \cdot 3(1/3)^2 + (1/3)(1/4)(3 \cdot 3 + 2 \cdot 5)} = \frac{135}{135 + 96 + 228} = \frac{135}{459} = \frac{5}{17}
\]
Problem 2 (continued)
Problem 3 (20 points)

Let $X$ be a continuous random variable with cumulative distribution function (CDF) shown in the diagram below.

(a) (10 points) Determine the probability that $\frac{2X-15}{5X-10}$ is positive.

SOLUTION:
Consider a more general CDF. The pdf of $X$ is shown to the right:

Since $f_X(u) = F'_X(u)$, we have $w = \frac{t}{s-r}$ and $v = \frac{1-t}{r-q}$. Suppose $q < d/c < r < b/a < s$. Then,

$$P \left( \frac{aX - b}{cX - d} > 0 \right) = P(aX - b > 0 \text{ and } cX - d > 0) + P(aX - b < 0 \text{ and } cX - d < 0)$$

$$= P(X > b/a \text{ and } X > d/c) + P(X < b/a \text{ and } X < d/c)$$

$$= v(s - (b/a)) + w((d/c) - q)$$

$$= \frac{(1-t)(s-b/a)}{s-r} + \frac{t(d/c-q)}{r-q}$$

$$= \frac{(1-t)(as-b)}{a(s-r)} + \frac{t(d-cq)}{c(r-q)}$$

$$= \frac{(1-(2/3))(2 \cdot 11 - 15)}{2(11-7)} + \frac{(2/3)(16 - 5 \cdot 2)}{5(7-2)} = \frac{7}{24} + \frac{4}{25} = \frac{271}{600}$$
Problem 3 (continued)

(b) (10 points) Find the expected value of $X$.

SOLUTION:

$$E[X] = \int_{-\infty}^{\infty} u f_X(u) du$$

$$= \int_{q}^{r} wudu + \int_{r}^{s} vudu$$

$$= \frac{w(r^2 - q^2)}{2} + \frac{v(s^2 - r^2)}{2}$$

$$= \frac{t(r^2 - q^2)}{2(r - q)} + \frac{(1 - t)(s^2 - r^2)}{2(s - r)}$$

$$= t \cdot \frac{r + q}{2} + (1 - t) \cdot \frac{s + r}{2}$$

$$= (2/3)^{7/2} + (1 - (2/3))(11/2) + 7 = (2/3)(9/2) + (1/3)(9) = \boxed{6}$$