INSTRUCTIONS
This exam is closed book and closed notes, with the one exception that you are allowed notes on both sides of one sheet of 8.5 x 11.0 inch paper (not tape or stick-ons allowed). No calculators, laptop computers, or other electronic devices are allowed.

Write your answers in the spaces provided. Show all your work. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress on a problem. Zero credit will be given for correct answers that lack adequate explanation of how they were obtained. Simplify your answers as much as possible and leave answers as fractions, not as decimal numbers.

GRADING
1a. 5 points
1b. 10 points
1c. 10 points
1d. 5 points
1e. 10 points
2. 10 points
TOTAL (50 points)
Problem 1 (40 points total)

Let $r, s \geq 2$. A red coin and a blue coin are randomly placed on a $r \times s$ rectangular grid in different locations. Here is an example where $r = 3$, $s = 5$, and the two coins are vertically adjacent:

(a) (5 points) Find the probability that the red coin is in the top row, given the blue coin is not in the top row. Do not assume $r = 3$ and $s = 5$ in this part. Your answer should involve $r$ and $s$.

SOLUTION:

Let $R$ and $B$ be the events that the red and blue coins are in the top row, respectively. There are $rs$ total squares, so there are $rs(rs - 1)$ different ways the red and blue coins can be placed in the grid. To count the number of these configurations where the red coin is in the top row, but the blue coin is not (i.e. the event $RB^c$), note there are $s$ squares for the red coin in the top row and $s(r - 1)$ squares for the blue coin not in the top row, so their product is the total number of configurations. To count the total number of ways the blue coin is not in the top row (i.e. the event $B^c$), there are $s(r - 1)$ squares to place the blue coin not in the top row, and then after such placement, the red coin can go in any one of the remaining $rs - 1$ squares, so multiplying them gives the total number. Thus,

$$P(R|B^c) = \frac{P(RB^c)}{P(B^c)} = \frac{s^2(r-1)}{s(r-1)(rs-1)} = \frac{s}{rs-1}.$$
(b) (10 points) Find the probability that the two coins are on squares that are adjacent either horizontally or vertically. If you cannot solve the problem for general \( r \) and \( s \), you may assume (for half credit only and for part (b) only) that \( r = s = 3 \).
Let us call the red coin the “first” coin and the blue coin the “second” coin. Let $C$ be the event that the first coin is placed on one of the 4 corner squares, each of which has 2 neighbors. Let $S$ be the event that the first coin is placed on one of the $2(r - 2) + 2(s - 2)$ non-corner side squares, each of which has 3 neighbors. Let $M$ be the event that the first coin is placed on one of the $(r - 2)(s - 2)$ middle squares, each of which has 4 neighbors. Let $A$ be the event that the second coin is in a square adjacent to the square containing the first coin.

$$P(A) = P(A|C)P(C) + P(A|S)P(S) + P(A|M)P(M)$$

$$= \frac{2}{rs - 1} \cdot \frac{4}{rs} + \frac{3}{rs - 1} \cdot \frac{2(r - 2) + 2(s - 2)}{rs} + \frac{4}{rs - 1} \cdot \frac{(r - 2)(s - 2)}{rs}$$

$$= \frac{2 \cdot 4 + 3(r + s - 4) + 2(r - 2)(s - 2)}{rs(rs - 1)}$$

$$= \frac{2(2rs - r - s)}{rs(rs - 1)}.$$ 

Alternatively, there are $\binom{rs}{2}$ ways of placing two coins on the $r \times s$ grid. Each of the 4 corner squares are part of 2 adjacent-square pairs. Each of the $2(r - 2) + 2(s - 2)$ non-corner side squares are part of 3 adjacent-square pairs. Each of the $(r - 2)(s - 2)$ middle squares are part of 4 adjacent-square pairs. But each of these adjacent square pairs is counted twice, so we get:

$$P(A) = \frac{1}{2} \cdot \frac{4 \cdot 2 + 3 \cdot (2(r - 2) + 2(s - 2)) + 4 \cdot (r - 2)(s - 2)}{\binom{rs}{2}} = \frac{2(2rs - r - s)}{rs(rs - 1)}.$$ 

If $r = s$, then

$$P(A) = \frac{2(2s^2 - 2s)}{s^2(s^2 - 1)} = \frac{4}{s(s + 1)}$$

so if $r = s = 3$, then $P(A) = 1/3$. 

Page 4 of 8
(c) (10 points) For this part, assume \( r = 3 \) and \( s = 6 \). Let \( X \) be the number of coins in the top row. Find and plot the probability mass function (pmf) of \( X \) and carefully label your plot.

**SOLUTION:**

The number of coins in the top row is either 0, 1, or 2. The probability of getting 0 coins in the top row is

\[
p_X(0) = \frac{s(r-1)}{\binom{rs}{2}} = \frac{(r-1)(rs-s-1)}{r(rs-1)}.
\]

The probability of getting 1 coin in the top row is

\[
p_X(1) = \frac{s \cdot s(r-1)}{\binom{rs}{2}} = \frac{2s(r-1)}{r(rs-1)}.
\]

The probability of getting 2 coins in the top row is

\[
p_X(2) = \frac{s^2}{\binom{rs}{2}} = \frac{s-1}{r(rs-1)}.
\]

Taking \( r = 3 \) and \( s = 6 \), we get

\[
\begin{align*}
p_X(0) &= \frac{22}{51} \\
p_X(1) &= \frac{24}{51} \\
p_X(2) &= \frac{5}{51}.
\end{align*}
\]
(d) (5 points) For this part, assume $r = 4$ and $s = 3$. Is the event that both coins lie in corner squares independent of the event that at least one coin does not lie in a corner square? Justify your answer.

**SOLUTION:**
Let $E$ be the event that both coins lie in corner squares. Then the event that at least one coin does not lie in a corner square is $E^c$. We have $P(EE^c) = P(\emptyset) = 0$. But $P(E) \neq 0$ and $P(E^c) \neq 0$, so $P(EE^c) \neq P(E)P(E^c)$. Thus, the events $E$ and $E^c$ are not independent.
(e) (10 points) For this part, assume \( r = 6 \) and \( s = 8 \). What is the probability that a coin showing Tails is in the top-left corner of the grid?

**SOLUTION:**
Let us call the red coin the “first” coin and the blue coin the “second” coin. Let \( E_n \) be the event that the \( n \)th coin is in the top-left corner, where \( n = 1, 2 \). Let \( H_n \) be the event that the \( n \)th coin shows Tails. Note that each \( E_n \) is physically independent of each \( H_n \). Then we get:

\[
P(E_1 H_1 \cup E_2 H_2) = P(E_1 H_1) + P(E_2 H_2) \\
= P(E_1)P(H_1) + P(E_2)P(H_2) \\
= (1/2)(P(E_1) + P(E_2)) \\
= (1/2)(P(E_1) + P(E_2|E_1)P(E_1) + P(E_2|E_1^c)P(E_1^c)) \\
= (1/2) \left( \frac{1}{rs} + 0 + \frac{1}{rs - 1} \left( 1 - \frac{1}{rs} \right) \right) \\
= \frac{1}{rs}
\]
Problem 2 (10 points)

Let $X$ be a continuous random variable with the property that for any two numbers $a, b$ satisfying $0 \leq a \leq b \leq 3$, the probability that $a < X < b$ is equal to $(b^2 - a^2)/9$. Determine and plot the probability density function (pdf) of $X$.

SOLUTION:

Notice that $P(0 < X < 3) = 1$, so all of the probability is located in the interval $[0, 3]$. Thus we may assume the pdf of $X$ is zero outside this interval. The CDF of $X$ in the interval $[0, 3]$ is

$$F_X(u) = P(X \leq u) = P(0 < X < u) = \frac{u^2}{9}.$$ 

Thus, the pdf of $X$ in the interval $[0, 3]$ is

$$f_X(u) = \frac{d}{du} \left( \frac{u^2}{9} \right) = \frac{2}{9}u$$

and zero elsewhere.