INSTRUCTIONS
This exam is open book and open notes. No calculators, laptop computers, or other electronic devices are allowed.

Write your answers in the spaces provided. Show all your work. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress on a problem. Zero credit will be given for correct answers that lack adequate explanation of how they were obtained. There is a maximum total of 40 points on this exam. Simplify your answers as much as possible and leave answers as fractions, not as decimal numbers.

GRADING

1. 10 points ______
2. 15 points ______
3. 15 points ______
TOTAL (40 points) ______
Problem 1 (10 points)
You are given a biased coin where the probabilities of Heads is $1/3$ and you flip the coin 9 times. What is the probability that you see at most 7 Heads, given that the first and the last flip are both Tails?

SOLUTION: 2 of the 9 flips are Tails, so of the remaining 7 flips, at most all of them could be Heads, so the probability that you see at most 7 Heads is 1.
Problem 2 (15 points)

The probability it is sunny is nonzero and the probability it rains is nonzero. If it’s sunny, then the probability it rains is $1/2$ times the probability it rains if it’s not sunny. On the other hand, the probability it is sunny is $3/2$ times the probability it is sunny given it rains. Find the probability it is sunny.

SOLUTION: Let $S$ and $R$ denote the events that it is sunny and it rains, respectively. For convenience, let $x = P[S]$ and $y = P[S|R]$. Since $P[S] = \beta P[S|R]$, we have $x = \beta y$. Thus,

$$\frac{yP[R]}{x} = \frac{P[S|R]P[R]}{P[S]} = P[R|S] = \alpha P[R|S^c] = \frac{\alpha P[S^c|R]P[R]}{P[S^c]} = \frac{\alpha (1 - y)P[R]}{1 - x}$$

$\therefore \frac{y}{x} = \alpha (1 - y)/(1 - x)$

$\therefore 1/\beta = \alpha (1 - y)/(1 - \beta y)$

$\therefore y = \frac{1 - \alpha \beta}{\beta (1 - \alpha)}$

$\therefore x = \frac{1 - \alpha \beta}{1 - \alpha}$

$\therefore = \frac{1 - (1/2)(3/2)}{1 - (1/2)}$

$= 1/2$
Problem 3 (15 points)

Let $X$ be a continuous random variable whose cumulative distribution function (CDF) $F_X(u)$ is $u(2-u)$ on the interval $[0, 1]$ and zero for $u < 0$ and one for $u > 1$. Find the probability that $X$ is larger than 9 times its variance.

SOLUTION: We have

\[
f_X(u) = \frac{d}{du}u(2-u) = 2(1-u) \quad \text{for} \quad 0 \leq u \leq 1
\]

\[
E[X] = \int_0^1 2u(1-u)du = 1/3
\]

\[
E[X^2] = \int_0^1 2u^2(1-u)du = 1/6
\]

\[
Var[X] = E[X^2] - (E[X])^2 = (1/6) - (1/3)^2 = 1/18
\]

\[
P[X > 9Var[X]] = \int_{1/2}^1 2(1-u)du = 1/4.
\]