INSTRUCTIONS
This exam is open book and open notes. No calculators, laptop computers, or other electronic devices are allowed.

Write your answers in the spaces provided. Show all your work. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress on a problem. Zero credit will be given for correct answers that lack adequate explanation of how they were obtained. There is a maximum total of 30 points on this exam. Simplify your answers as much as possible and leave answers as fractions, not decimal numbers.

GRADING

1. 10 points
2. 10 points
3. 10 points
TOTAL (30 points)
Problem 1 (10 points)
You have 3 biased coins. The probability of getting a Heads on the blue coin is $b$, on the red coin is $r$, and on the green coin is $g$. First you flip the blue coin once. If it comes up Heads then you flip the red coin once, otherwise you flip the green coin once. So you have flipped a total of 2 coins.

What is the probability that both coin flips come up Heads?

SOLUTION: Let $A$ denote the event that the first coin flip came up Heads, and $B$ denote the event that the second coin flip came up Heads. We are given:

\[
P[A] = b \\
P[B|A] = r \\
P[B|A^c] = g.
\]

We need to find $P[AB]$.
We have

\[
\]
If the second coin flip comes up Heads, what is the probability that the first coin flip was Heads?

**SOLUTION:** We need to find $P[A|B]$. We have

\[
\]
If at least one of the two coin flips is Heads, what is the probability that both of the coin flips were Heads?

**SOLUTION:** We need to find $P[AB|A \cup B]$. We have

$$P[AB|A \cup B] = \frac{P[AB(A \cup B)]}{P[A \cup B]}$$

$$= \frac{P[AB]}{P[A \cup B]}$$

$$= \frac{P[AB]}{P[A]P[B|A] + P[B|A^c]}$$

$$= \frac{P[AB]}{P[A] + P[A^c]P[B|A^c]}$$

$$= \frac{b}{b + (1 - b)g}.$$
Problem 2 (10 points)
Find the expected value $E[X]$ of a random variable $X$ which has cumulative distribution function (CDF) given by

$$F_X(u) = \begin{cases} 
1 - 5^{-n} & \text{when } u \in [3^{n-1}, 3^n) \text{ where } n = 1, 2, 3, \ldots \\
0 & \text{else.}
\end{cases}$$

SOLUTION: The random variable $X$ is clearly discrete. Its probability mass function (pmf) $p_X(u)$ is zero everywhere except at points $u = 1, 3, 9, 27, 81, \ldots$. So, In the definition of the CDF, the function $F_X(u)$ is defined in pieces, namely in the intervals $[1, 3), [3, 9), [9, 27), \ldots$ etc. and then also in $(-\infty, 1)$. At the point $u = 1$, since $1 \in [1, 3)$ we have $F_X(1) = 1 - 5^{-1} = 4/5$. Thus, $p_X(1) = F_X(1) - F_X(1^-) = 4/5 - 0 = 4/5$ is a special case.

For $n = 2, 3, 4, \ldots$,

$$p_X(3^{n-1}) = F_X(3^{n-1}) - F_X((3^{n-1})^-)$$
$$= F_X(3^{n-1}) - F_X(3^{n-2})$$
$$= 1 - 5^{-(n-1)}$$
$$= (1 - 5^{-n}) - (1 - 5^{-(n-1)})$$
$$= 5^{-(n-1)} - 5^{-n}$$
$$= 5^{-n}(5 - 1)$$
$$= 4 \cdot 5^{-n}$$

$$p_X(1) = F_X(1) - F_X(1^-) = 4/5 - 0 = 4/5$$

$$E[X] = \sum_{n=1}^{\infty} 3^{n-1} p_X(3^{n-1})$$
$$= 1 \cdot p_X(1) + \sum_{n=2}^{\infty} 3^{n-1} \cdot 4 \cdot 5^{-n}$$
$$= (4/5) + 4 \sum_{n=2}^{\infty} 3^{n-1} 5^{-n}$$
$$= (4/5) + (4/3) \sum_{n=2}^{\infty} (3/5)^n$$
$$= (4/5) + (4/3) \cdot \frac{(3/5)^2}{1 - (3/5)}$$
$$= (4/5) + \frac{12/25}{2/5}$$
$$= (4/5) + 6/5$$
$$= 2$$
Problem 3 (10 points)

Let $X$ be a continuous random variable whose probability density function (pdf) is:

$$f_X(u) = \begin{cases} 
3e^{-3u} & \text{if } u > 0 \\
0 & \text{else}.
\end{cases}$$

Find a nonnegative real number $A$ such that the event $\{X > A\}$ is independent of the event $\{X < 7\}$.

**SOLUTION:** Consider the more general pdf $f(u) = \lambda e^{-\lambda u}$ for $u > 0$, and denote the latter event by $\{X < C\}$. Clearly we must have $A < C$, for otherwise $P(X > A|X < C) = 0 \neq P(X > A)$. Thus, the events are independent if

$$P(A < X < C) = P(X > A)P(X < C).$$

Equivalently, we get independence if

$$\int_A^C \lambda e^{-\lambda u} du = \left(\int_A^\infty \lambda e^{-\lambda u} du\right) \left(\int_0^C \lambda e^{-\lambda u} du\right).$$

$$\therefore e^{-\lambda A} - e^{-\lambda C} = e^{-\lambda A}(1 - e^{-\lambda C})$$

$$\therefore e^{-\lambda A} = 1$$

$$\therefore A = 0.$$