Name ________________________________

Your UCSD ID Number ________________________________

Signature ________________________________

INSTRUCTIONS
This exam is closed book and closed notes, with the one exception that you are allowed notes on both sides of one sheet of 8.5 x 11.0 inch paper (not tape or stick-ons allowed). No calculators, laptop computers, or other electronic devices are allowed.

Write your answers in the spaces provided. Show all your work. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress on a problem. Zero credit will be given for correct answers that lack adequate explanation of how they were obtained. Simplify your answers as much as possible and leave answers as fractions, not as decimal numbers.

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<th>GRADING</th>
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<td>1. 10 points _____</td>
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<td>4(a). 10 points _____</td>
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Problem 1 (10 points)

Suppose you toss a fair coin 3 times and $X$ is the number of Heads that come up. Determine the variance of $4\sqrt{X} - 7$.

**SOLUTION:** Recall (or recompute), that if $A$ and $B$ are constants, then

$$\text{Var}(Y + B) = E[(Y + B)^2] - (E[Y + B])^2 = E[Y^2] - (E[Y])^2 = \text{Var}(Y)$$


If a coin is tossed 3 times, we get a binomial pmf. Thus,

$$P_X(k) = \binom{3}{k} (1/2)^k (1/2)^{3-k} = \frac{1}{8} \binom{3}{k}$$

$$= \begin{cases} 
1/8 & \text{if } k = 0 \\
3/8 & \text{if } k = 1 \\
3/8 & \text{if } k = 2 \\
1/8 & \text{if } k = 3 
\end{cases}$$

$$\text{Var}(A\sqrt{X} + B) = \text{Var}(A\sqrt{X})$$

$$= A^2\text{Var}(\sqrt{X})$$

$$= A^2 \left( E[(\sqrt{X})^2] - (E[\sqrt{X}])^2 \right)$$

$$= A^2 \left( E[X] - (E[\sqrt{X}])^2 \right)$$

$$= A^2 \left( \frac{3}{2} - \left( \frac{3}{8} + \frac{3}{8} \cdot \sqrt{2} + \frac{1}{8} \cdot \sqrt{3} \right)^2 \right)$$

$$= A^2 \left( \frac{3}{2} - \frac{1}{64} \left( 3 + 3\sqrt{2} + \sqrt{3} \right)^2 \right)$$

$$= \frac{3A^2}{32} \left( 11 - 3\sqrt{2} - \sqrt{3} - \sqrt{6} \right)$$
Problem 2 (10 points)
Let $X$ be a Gaussian random variable with mean $3$ and variance $9$. Find the expected value of $e^X$.

SOLUTION: Suppose $X \sim N(m, \sigma^2)$. Then,

$$E[e^X] = \int_{-\infty}^{\infty} e^u \cdot \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{u-m}{\sigma} \right)^2} \, du$$

(let $v = \frac{u-m}{\sigma}$, $dv = du/\sigma$)

$$= \frac{1}{\sigma \sqrt{2\pi}} \int_{-\infty}^{\infty} e^{m+\sigma v} e^{-v^2/2} \, (dv)$$

$$= \frac{e^m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\sigma v - (v^2/2)} \, dv$$

$$= e^{m+(\sigma^2/2)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2} (v-\sigma)^2} \, dv$$

$$= e^{m+(\sigma^2/2)}$$
Problem 3 (10 points)
Suppose $X$ is a continuous random variable uniformly distributed in the interval $[4, 8]$ and let $Y = \frac{1}{X+a}$. Find and plot the probability density function of $Y$.

SOLUTION: Suppose $X$ is uniform on $[b, c]$ and $Y = \frac{1}{X+a}$, where $0 < b < c$. Clearly, $P(X \leq u) = 0$ if $u < 0$. So for $u > 0$,

$$F_Y(u) = P\left(\frac{1}{X+a} \leq u\right) = P(X + a \geq 1/u) = P(X \geq (1/u) - a)$$

$$= 1 - P(X \leq (1/u) - a) = 1 - F_X((1/u) - a)$$

$$f_Y(u) = \frac{d}{du} (1 - F_X((1/u) - a)) = -f_X((1/u) - a) \cdot \frac{d}{du} \left(\frac{1}{u} - a\right) = \frac{1}{u^2} f_X\left(\frac{1}{u} - a\right)$$

Since $X$ is uniform on $[b, c]$, we know that $f_X\left(\frac{1}{u} - a\right) = \frac{1}{c-b}$ when $b \leq \frac{1}{u} - a \leq c$ and is zero otherwise. Equivalently, $f_X$ is nonzero precisely when $\frac{1}{a+c} \leq u \leq \frac{1}{a+b}$. So we get,

$$f_Y(u) = \begin{cases} \frac{1}{u^2(c-b)} & \text{if } \frac{1}{a+c} \leq u \leq \frac{1}{a+b} \\ 0 & \text{else} \end{cases}$$
Problem 4 (30 points)

Suppose $X$ and $Y$ are random variables whose joint probability density function $f_{X,Y}(u, v)$ is uniform on the shaded region shown in the figure.

(a) (10 points) Determine the marginal probability density function of $X$.

SOLUTION: There are 15 unit squares in the shaded region, so the joint density in the shaded region equals $1/15$. The marginal pdf of $X$ is $f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u, v) dv$, which is an integral along vertical slices located at horizontal position $u$. If $u \in [1, 2]$, then $f_X(u)$ is an integral of a constant function of total length 5 (i.e. on the vertical interval $[1, 6]$). If $u \in [2, 3]$, then $f_X(u)$ is an integral of a constant function of total length 2 (i.e. on the vertical intervals $[1, 2]$ and $[5, 6]$). If $u \in [3, 4]$, then $f_X(u)$ is an integral of a constant function of total length 3 (i.e. on the vertical intervals $[1, 2]$, $[3, 4]$, and $[5, 6]$). If $u \in [4, 5]$, then $f_X(u)$ is an integral of a constant function of total length 4 (i.e. on the vertical intervals $[1, 4]$ and $[5, 6]$). If $u \in [5, 6]$, then $f_X(u)$ is an integral of a constant function of total length 1 (i.e. on the vertical interval $[5, 6]$). Thus we have:

$$f_X(u) = \frac{1}{15} \begin{cases} 
5 & \text{if } 1 \leq u < 2 \\
2 & \text{if } 2 \leq u < 3 \\
3 & \text{if } 3 \leq u < 4 \\
4 & \text{if } 4 \leq u < 5 \\
1 & \text{if } 5 \leq u < 6 \\
0 & \text{else}
\end{cases}$$
(b) (10 points) Determine $F_{X,Y}(\pi, \pi)$, where $F_{X,Y}$ is the joint cumulative distribution function of $X$ and $Y$.

**SOLUTION:** $F_{X,Y}(\pi, \pi) = P(X \leq \pi, Y \leq \pi)$. As shown in the figure below, the probability consists of adding up the probabilities contributed by the squares $[1, 2]^2$ (area 1), $[2, 3] \times [1, 2]$ (area 1), $[1, 2] \times [2, 3]$ (area 1), $[3, \pi]^2$ (area $(\pi - 3)^2$), and the rectangles $[3, \pi] \times [1, 2]$ (area $\pi - 3$), $[1, 2] \times [3, \pi]$ (area $\pi - 3$). The total probability is thus $F_{X,Y}(\pi, \pi) = \frac{3 + (\pi - 3)^2 + 2(\pi - 3)}{15} = \frac{\pi^2 - 4\pi + 6}{15}$. 

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(c) (10 points) Determine the probability that $X^2 + 4$ is less than $Y$.

**SOLUTION:** The desired probability consists of the double integral of the joint density function above the parabola $v = u^2 + 4$, as shown in the diagram. The parabola intersects the boundary of the shaded region at $(1, 5)$ and $(\sqrt{2}, 6)$ (the latter point is obtained by solving $u^2 + 4 = 6$). We will integrate the constant joint density (i.e. $1/15$) over the portion of the square $[1, 2] \times [5, 6]$ that lies above the parabola, by using vertical strips.
\[ P(X^2 + 4 < Y) = P((X, Y) \in T) \]

(\text{where } T = \{(u, v) \in \mathbb{R}^2 : v > u^2 + 4\})

\[ = \int_{u^2+4}^{\sqrt{2}} \int_{-\infty}^{6} (1/15) dv du \]

\[ = (1/15) \int_{1}^{\sqrt{2}} (2 - u^2) du \]

\[ = (1/15)(2u - (1/3)u^3) \bigg|_{1}^{\sqrt{2}} \]

\[ = (1/15) \left( (2\sqrt{2} - (2/3)\sqrt{2}) - (2 - (1/3)) \right) \]

\[ = (1/15) \left( (4/3)\sqrt{2} - (5/3) \right) \]

\[ = \frac{4\sqrt{2} - 5}{45} \]