INSTRUCTIONS
This exam is closed book and closed notes, with the one exception that you are allowed notes on both sides of one sheet of 8.5 x 11.0 inch paper (not tape or stick-ons allowed). No calculators, laptop computers, or other electronic devices are allowed.

Write your answers in the spaces provided. Show all your work. If you need extra space, please use the back of the previous page. Partial credit will be given only for substantial progress on a problem. Zero credit will be given for correct answers that lack adequate explanation of how they were obtained. Simplify your answers as much as possible and leave answers as fractions, not as decimal numbers.
Problem 1 (10 points)

For three events \( A, B, \) and \( C, \) we know that:

- \( A \) and \( C \) are independent,
- \( B \) and \( C \) are independent,
- \( A \) and \( B \) are disjoint,
- \( P(A \cup C) = \frac{5}{8}, \) \( P(B \cup C) = \frac{7}{12}, \) \( P(A \cup B \cup C) = \frac{17}{24}. \)

Find \( P(A), P(B), \) and \( P(C). \)

SOLUTION: Let \( P(A) = a, \) \( P(B) = b, \) and \( P(C) = c. \) By independence, we have \( P(AC) = P(A)P(C) = ac \) and \( P(BC) = P(B)P(C) = bc. \)

\[
P(A \cup C) = a + c - ac = \frac{5}{8}
\]
\[
P(B \cup C) = b + c - bc = \frac{7}{12}
\]
\[
P(A \cup B \cup C) = a + b + c - ac - bc = \frac{17}{24}.
\]

By subtracting the third equation from the sum of the first and second equations, we get \( c = (\frac{5}{8}) + (\frac{7}{12}) - (\frac{17}{24}) = \frac{1}{2}. \) Plugging \( c \) into the first equation then gives \( a = \frac{1}{4}, \) and plugging \( c \) into the second equation gives \( b = \frac{1}{6}. \)
Problem 1 (continued)
Problem 2 (10 points)

A box contains three coins: two regular coins and one fake two-Headed coin (i.e. no Tails). The regular coins are each biased such that the probability of Heads is $\frac{1}{5}$ and the probability of Tails is $\frac{4}{5}$. You pick one of the three coins at random and toss it, and get Heads. What is the probability that it is the two-Headed coin?

SOLUTION:
Let $H$ be the event the chosen coin lands Heads up. Let $R$ be the event that you choose a regular coin. Then $R^c$ is the event that you choose the two-Headed coin. Thus,

$$P(H|R) = q$$
$$P(H|R^c) = 1$$

$$P(H) = P(H|R)P(R) + P(H|R^c)P(R^c)$$
$$= q \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}$$
$$= \frac{2q + 1}{3}.$$  

$$P(R^c|H) = \frac{P(H|R^c)P(R^c)}{P(H)}$$
$$= \frac{1 \cdot \frac{1}{3}}{\frac{2q + 1}{3}}$$
$$= \frac{1}{2q + 1}.$$  

If $q = 1/5$, then $P(R^c|H) = 5/7$. 

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Problem 2 (continued)
Problem 3 (10 points total)

(a) (5 points) Let $X$ be a discrete random variable whose probability mass function (pmf) is given by:

$$p_X(k) = \frac{A}{B^k} \quad k = 1, 2, 3, \ldots$$

where $A$ and $B$ are positive constants. Suppose the probability that $X$ is 3 or more is $2/3$. Determine the values of $A$ and $B$.

SOLUTION:

$$1 = \sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} \frac{A}{B^k} = A \left( \frac{1/B}{1-1/B} \right) = \frac{A}{B-1}$$

∴ $B = A + 1$

$$c = P(X < 3) = 1 - P(X \geq 3) = p_X(1) + p_X(2) = \frac{B-1}{B} + \frac{B-1}{B^2} = 1 - \frac{1}{B^2}$$

∴ $B = \sqrt{1 - c}$

If $c = 1/3$, then $B = \sqrt{3/2}$ and $A = \sqrt{3/2} - 1$. 
(b) (5 points) Find the probability that $X$ is greater than 2, given that $X$ is greater than 1. Leave your answer in terms of $A$ and $B$ (i.e. don’t substitute in the values for $A$ and $B$ that you found in part (a)).

**SOLUTION:**

$$P(X > 2 | X > 1) = \frac{P(X > 2, X > 1)}{P(X > 1)} = \frac{P(X > 2)}{P(X > 1)} = \frac{1 - p_X(1) - p_X(2)}{1 - p_X(1)}$$

$$= 1 - \frac{p_X(2)}{1 - p_X(1)} = 1 - \frac{A/B^2}{1 - (A/B)}$$

$$= 1 - \frac{A}{B(B - A)}$$
Problem 4 (10 points)

Let $X$ be a continuous random variable whose probability density function (pdf) is given by:

$$f_X(u) = \begin{cases} 
3e^{-3u} & \text{if } u > 0 \\
0 & \text{else}
\end{cases}$$

Find the probability that $\frac{2X+4}{3X+4}$ is greater than 2.

SOLUTION:
A more general solution is given below, where it is assumed that $a, b, c, d, e > 0$, $P(X < 0) = 0$, and $a > ce$:

$$P\left(\frac{aX + b}{cX + d} > e\right) = P(X(a - ce) > ed - b)$$

$$= P\left(X > \frac{ed - b}{a - ce}\right)$$

$$= \int_{\frac{ed - b}{a - ce}}^{\infty} ke^{-ku} du$$

$$= e^{-k\left(\frac{ed - b}{a - ce}\right)}$$

If $a = 9$, $b = 4$, $c = 3$, $d = 4$, $e = 2$, $k = 3$, then the probability is $e^{-4}$. 