YOU MUST SHOW YOUR WORK TO RECEIVE CREDIT. A CORRECT ANSWER WITHOUT SHOWING YOUR REASONING WILL NOT RECEIVE CREDIT.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points Possible</th>
<th>Points Earned</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td></td>
</tr>
</tbody>
</table>
Problem 1. An instructor gives her class a set of 20 problems with the information the final exam will consist of a random selection of 10 of them. If a student has figured out how to do 12 of the problems, what is the probability that he or she will correctly answer

(a) all 10 problems on the exam;

(b) at least 8 of the problems on the exam.

Solution. Imagine that an urn is filled with 20 balls, each representing one of the possible exam problems. Those balls corresponding to problems studied by the student are colored red; there are 12 such balls. The remaining balls are colored black. The instructor draws 10 of these balls at random without replacement. [Obviously, an instructor would not put the same problem twice on the exam! Thus, he samples without replacement.] Let \( X \) be the number of red balls drawn; this corresponds to the number of questions which the student can answer correctly.

(a)

\[
P[X = 10] = \binom{12}{10} \binom{8}{0} \binom{20}{10}.
\]

(b)

\[
P[X \geq 8] = P[X = 8] + P[X = 9] + P[X = 10] = \binom{12}{8} \binom{8}{2} \binom{20}{10} + \binom{12}{9} \binom{8}{1} \binom{20}{10} + \binom{12}{10} \binom{8}{0} \binom{20}{10}.
\]
**Problem 2.** Ninety-eight percent of all babies survive delivery. However, 15 percent of all births involve Cesarean (C) sections, and when a C section is performed the baby survives 96 percent of the time. If a randomly chosen pregnant woman does not have a C section, what is the probability that her baby survives?

**Solution.** Let \( S \) be the event of survival, and \( C \) be the probability of Cesarian. We want to find \( P(S \mid C') \), and are given that \( P(S) = 0.98 \), \( P(S \mid C) = 0.96 \), and \( P(C) = 0.15 \). The definition of conditional probability gives us

\[
P(S \mid C') = \frac{P(S \cap C')}{P(C')}.
\]

We can find \( P(S \cap C') \) from the information given by using the identity

\[
P(S) = P(S \cap C) + P(S \cap C').
\]

In particular, rearranging and then using the conditional probability identity \( P(S \cap C) = P(S \mid C)P(C) \) yields

\[
P(S \cap C') = P(S) - P(S \cap C)
= P(S) - P(S \mid C)P(C)
= 0.98 - (0.96)(0.15)
= 0.836
\]

Since \( P(C') = 1 - P(C) = 0.85 \), we have then

\[
P(S \mid C') = \frac{0.836}{0.85} \approx 0.9835.
\]

\(\square\)
Problem 3. Urn $A$ has 5 white and 7 black balls. Urn $B$ has 3 white and 12 black balls. We flip a fair coin. If the outcome is heads, then a ball from urn $A$ is selected, whereas if the outcome is tails, then a ball from urn $B$ is selected. Suppose that a white ball is selected. What is the probability that the coin landed tails?

Solution.

\[
P(T \mid W) = \frac{P(T \cap W)}{P(W)} = \frac{P(W \mid H)P(H)}{P(W \mid H)P(H) + P(W \mid T)P(T)}
\]

\[
= \frac{\frac{3}{15} \cdot \frac{1}{2}}{\frac{5}{12} \cdot \frac{1}{2} + \frac{3}{15} \cdot \frac{1}{2}}
\]

\[
= \frac{12}{37}.
\]
**Problem 4.** Suppose that $X$ has the *distribution function*

\[
F(t) = \begin{cases} 
0 & \text{if } t < 1 \\ 
1 - \frac{1}{t^3} & \text{if } t \geq 1.
\end{cases}
\]

Find $E(X)$.

**Solution.** The density is given by differentiating $F$:

\[
f(t) = \begin{cases} 
0 & \text{if } t < 1 \\ 
3t^{-4} & \text{if } t > 1.
\end{cases}
\]

Thus,

\[
E(X) = \int_{-\infty}^{\infty} tf(t)\,dt = \int_{1}^{\infty} t(3t^{-4})\,dt = \left[ -\frac{3t^{-2}}{2} \right]_{1}^{\infty} = \frac{3}{2}.
\]
Problem 5. Find the probability density function of $R = A\sin \theta$, where $A$ is a constant and $\theta$ is uniformly distributed on $(-\pi/2, \pi/2)$.

*Hint:* The derivative of $\arcsin(t)$ is $\frac{1}{\sqrt{1-t^2}}$.

**Solution.**

*Instructor’s Note:* The derivative of $\arcsin$ was incorrectly given on the test. If you used this misinformation, you were not penalized!

Notice that $\sin \theta$ is one-to-one on $(-\pi/2, \pi/2)$:

![Graph of sin(θ)](image)

Clearly then $A\sin \theta$ is also one-to-one, but it ranges from $-A$ to $A$. For $-A \leq t \leq A$

$$F_R(t) = P[R \leq t] = P[A\sin \theta \leq t] = P[\sin \theta \leq t/A] = P[\theta \leq \arcsin(t/A)] = \frac{\arcsin(t/A) + \pi/2}{\pi}$$

We conclude that

$$F_R(t) = \begin{cases} 
0 & t < -\pi/2 \\
\frac{\arcsin(t/A) + \pi/2}{\pi} & -\pi/2 \leq t \leq \pi/2 \\
1 & t > \pi/2.
\end{cases}$$

To get the pdf, we differentiate:

$$f_R(t) = \begin{cases} 
0 & t < -\pi/2 \\
\frac{1}{\pi A \sqrt{1-(t/A)^2}} & -\pi/2 \leq t \leq \pi/2 \\
0 & t > \pi/2.
\end{cases}$$
Problem 6. Let $X$ and $Y$ have joint pdf

$$f(s, t) = \begin{cases} se^{-s(t+1)} & \text{if } s > 0, t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the conditional probability density function of $Y$ given $X = t$.
(b) Find the density function of $Z = XY$.

Solution.

$$f_X(s) = \int_0^\infty se^{-s(t+1)} \, dt = -e^{-s(t+1)} \bigg|_0^\infty = e^{-s}.$$  Consequently, if $s > 0$ and $t > 0$,

$$f_{Y|X}(t \mid s) = \frac{f(s, t)}{f_X(s)} = \frac{se^{-s(t+1)}}{e^{-s}} = se^{-st}.$$  That is, given $X = s$, the conditional distribution of $Y$ is exponential with parameter $s$.

We compute the density of $Z$ in two ways. For $u > 0$

$$F_Z(u) = P[XY \leq u] = \int_0^\infty \int_0^{u/s} se^{-s(t+1)} \, dt \, ds = \int_0^\infty e^{-s} \int_0^{u/s} se^{-st} \, dt \, ds = \int_0^\infty e^{-s} \left[ e^{-st} \right]_0^{u/s} \, ds = \int_0^\infty e^{-s} \left[ 1 - e^{-u} \right] \, ds = \left[ 1 - e^{-u} \right].$$  Differentiating,

$$f_Z(u) = \begin{cases} e^{-u} & \text{if } u \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$  We can also compute as follows:

$$P[XY \leq u] = \int_0^\infty P[Y \leq u/s \mid X = s] e^{-s} \, ds = \int_0^\infty (1 - e^{-u}) e^{-s} \, ds = (1 - e^{-u}) \int_0^\infty e^{-s} \, ds = (1 - e^{-u}).$$  The second inequality follows from part (a).
Problem 7. 12 people get on an elevator on the ground floor of a 10 story building. Each person selects a floor; assume that each person selects independently and each person picks one of the 10 possible floors uniformly at random. No new people get on the elevator after the ground floor. Compute the expected number of stops the elevator makes.

Solution. Let $X$ be the number of stops the elevator makes. We can write $X = \sum_{i=1}^{10} I_i$, where

$$I_i = \begin{cases} 1 & \text{if the elevator stops at floor } i, \\ 0 & \text{otherwise}. \end{cases}$$

Since expectation is linear, we have

$$E(X) = E\left(\sum_{i=1}^{10} I_i\right) = \sum_{i=1}^{10} E(I_i) = \sum_{i=1}^{10} P\{\text{stop at floor } i\}. $$

Now

$$P\{\text{stop at floor } i\} = 1 - P\{\text{no-one picks floor } i\} = 1 - \left(\frac{9}{10}\right)^{12}. $$

So

$$E(X) = 10 \left[ 1 - \left(\frac{9}{10}\right)^{12} \right].$$

$\square$
Problem 8. Suppose two roommates, Bill and Bob, share a phone line. The number of calls received during the evening is a Poisson random variable with parameter 10. Bill is more popular, so each call independently has probability 0.7 of being for Bill.

(a) Find the conditional pmf for $X$, the number of calls received by Bill, given that the total number of calls is $n$.

(b) Find the pmf for $X$.

(c) Given that Bill receives 5 calls, what is the probability that Bob receives exactly one call.

Solution. Let $X$ be the number of calls for Bill, and $Y$ the number of calls for Bob. Let $Z = X + Y$. Given that $Z = n$, each of the $n$ call is for Bill with probability 0.7, and not for Bill with probability 0.3. The calls are independent of each other. Thus the conditional distribution of $X$ is Binomial$(n, p = 0.7)$.

Then

$$p_X(k) = \sum_{n=k}^{\infty} p_{X|Z}(k \mid n) p_Z(n) = \sum_{n=k}^{\infty} \binom{n}{k} (0.7)^k (0.3)^{n-k} e^{-10} \frac{(10)^n}{n!}$$

$$= \frac{e^{-10} (0.7)^k (10)^k}{k!} \sum_{n=k}^{\infty} \frac{(0.3)^{n-k} (10)^{n-k}}{(n-k)!} = \frac{e^{-10} 7^k}{k!} e^3 = \frac{e^{-7} 7^k}{k!}.$$

Thus $X$ has a Poisson(7) distribution. Similarly, $Y$ has a Poisson(3) distribution.

We want to find $P\{Y = 1 \mid X = 5\}$.

$$P\{Y = 1 \mid X = 5\} = \frac{P\{X = 5, Y = 1\}}{P\{X = 5\}} = \frac{P\{X = 5\} P\{Z = 6\}}{P\{X = 5\}} = \frac{\binom{6}{5} (0.7)^5 (0.3)^{10-5} e^{-10}}{6!}$$

$$= \frac{e^{-7} 7^5}{5!} = 3e^{-3}. $$
**Problem 9.** A column bet in roulette wins 2$ with probability 12/38, and losses 1$ with probability 26/38.

(a) Compute the mean and standard deviation of your winnings on a single game.

(b) You place this bet 25 times. Estimate the probability that you have won a positive amount.

(c) You place this bet 1000 times. Estimate the probability that you have won a positive amount.

**Solution.** Suppose that $X_i$ is the amount won on the $i$th game. $X_1, X_2, \ldots$ are independent and all have the same distribution. Then

$$E(X_1) = 2 \frac{12}{38} - \frac{26}{38} = -\frac{1}{19} \approx -0.05263.$$ 

and

$$E(X_1^2) = 2^2 \frac{12}{38} + 1^2 \frac{26}{38} = \frac{37}{19} \approx 1.947.$$ 

Thus

$$V(X_1) = E(X_1^2) - [E(X_1)]^2 = \frac{37}{19} - \frac{1}{361} = \frac{702}{361} \approx 1.945.$$ 

and so $SD(X_1) \approx 1.394$.

Let $S_n = \sum_{i=1}^n X_i$. Write $\mu$ for $E(X_1)$ and $\sigma$ for $SD(X_1)$.

$$P[S_{25} > 0] = P \left\{ \frac{S_{25} - 25\mu}{5\sigma} > \frac{0 - 25(-0.0526)}{5(1.394)} \right\}$$

$$= P \left\{ \frac{S_{25} - 25\mu}{5\sigma} > 1.316 \right\}$$

$$= P \left\{ \frac{S_{25} - 25\mu}{5\sigma} > 0.189 \right\}$$

$$\approx 1 - \Phi(0.189)$$

$$= 0.425$$

$$P[S_{1000} > 0] = P \left\{ \frac{S_{1000} - 1000\mu}{\sqrt{1000}\sigma} > \frac{52.63}{44.08} \right\}$$

$$= P \left\{ \frac{S_{25} - 1000\mu}{\sqrt{1000}\sigma} > 1.194 \right\}$$

$$\approx 1 - \Phi(1.194)$$

$$= 0.116$$
Problem 10. Let $X$ have a Gamma($\alpha, \lambda$) distribution, and let $Y$ be an independent Gamma($\beta, \lambda$) random variable. Let $Z = X + Y$.

(a) Find the MGF of $Z$. You can use Table 7.2.

(b) What is the distribution of $Z$?

Solution.

$$M_Z(t) = M_X(t)M_Y(t)$$

$$= \left( \frac{\lambda}{\lambda - t} \right)^{\alpha} \left( \frac{\lambda}{\lambda - t} \right)^{\beta}$$

$$= \left( \frac{\lambda}{\lambda - t} \right)^{\alpha + \beta}$$

Using Table 7.2, we see that $Z$ has a Gamma($\alpha + \beta, \lambda$) distribution.