Problem 4.1
Suppose we flip a coin. If the coin is heads, we randomly pick an integer from \{-2, -1, 0, 1, 2\}. If the coin is tails, we randomly pick an integer from \{0, 1, 2, 3, 4\}. Let \(X\) be the integer that we pick.

(a) What is the PMF of \(X\)?

(b) What is the expected value of \(X\)?

(c) What is the probability the coin was heads, assuming the number we picked was 2?

(d) Suppose \(Y = X^2\). What is the PMF of \(Y\)?

Solutions

(a) Let \(H\) be the event the coin was heads, then

\[
P(X = k \mid H) = \begin{cases} 1/5 & \text{if } k = -2, -1, 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}
\]

and

\[
P(X = k \mid H^c) = \begin{cases} 1/5 & \text{if } k = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}
\]

so

\[
P(X = k) = P(X = k \mid H) P(H) + P(X = k \mid H^c) P(H^c) = \begin{cases} 1/10 & \text{if } k = -2, -1 \\ 1/5 & \text{if } k = 0, 1, 2 \\ 1/10 & \text{if } k = 3, 4 \\ 0 & \text{otherwise} \end{cases}
\]

To verify that this is valid, we note that \(\sum_{k=-2}^{4} P(X = k) = 1\).

(b) We have

\[
E[X] = \sum_{k=-2}^{4} k P(X = k) = \frac{-2}{10} + \frac{-1}{10} + \frac{1}{5} + \frac{2}{5} + \frac{3}{10} + \frac{4}{10} = 1
\]

(c)

\[
P(H \mid X = 2) = \frac{P(X = 2 \mid H) P(H)}{P(X = 2)} = \frac{(1/5)(1/2)}{(1/5)} = 1/2
\]

which implies \(H\) and \(\{X = 2\}\) are independent events. However, if we know that \(X = 3\), then it must be the case that the coin was tails, so \(P(H \mid X = 3) = 0\), so the random variable \(X\) and the coin are clearly not independent in general.

(d) Clearly \(P(Y = k) = 0\) when \(k < 0\), since \(Y = X^2\). For \(k \geq 0\), we have

\[
P(Y = k) = P(X^2 = k) = P(X = \sqrt{k}) + P(X = -\sqrt{k}) = \begin{cases} 1/5 & \text{if } k = 0 \\ 3/10 & \text{if } k = 1, 4 \\ 1/10 & \text{if } k = 9, 16 \end{cases}
\]
Problem 4.2

Suppose the PMF of a random variable \( X \) is

\[
P(X = k) = \begin{cases} 
  c^k & \text{if } k = 1, 2, \ldots \\
  0 & \text{otherwise.}
\end{cases}
\]

(a) What value should \( c \) take on?

(b) Let \( Y = \sin(X \pi/2) \). What is the PMF of \( Y \)?

(c) What is the probability that \( P(X^2 + 2X > 8) \)?

(d) For what value of \( m \) do we have \( P(X \geq m) = 1/2 \)?

Solutions

(a) We have

\[
1 = \sum_{k} P(X = k) = \sum_{k=1}^{\infty} c^k = \frac{c}{1-c}
\]

which implies \( c = 1/2 \).

(b) We have \( X \in \{1, 2, 3, 4, 5, 6, \ldots \} \).

When \( X = 1, 5, 9, 13, \ldots, 4n + 1, \ldots \), we have \( Y = \sin(\pi/2) = 1 \), so

\[
P(Y = 1) = \sum_{n=0}^{\infty} P(X = 4n + 1) = \sum_{n=0}^{\infty} (1/2)^{4n+1} = \frac{1}{2} \sum_{n=0}^{\infty} (1/16)^n = \frac{1}{2} \frac{1}{1-1/16} = \frac{8}{15}
\]

When \( X = 2, 4, 6, 8, 10, \ldots, 2n, \ldots \), we have \( Y = \sin(\pi) = 0 \), so

\[
P(Y = 0) = \sum_{n=1}^{\infty} P(X = 2n) = \sum_{n=1}^{\infty} (1/2)^{2n} = \frac{1}{4} \sum_{n=1}^{\infty} (1/4)^n = \frac{1}{4} \frac{1}{1-1/4} = \frac{1}{3}
\]

When \( X = 3, 7, 11, 15, \ldots, 4n + 3, \ldots \), we have \( Y = \sin(3\pi/2) = -1 \), so

\[
P(Y = -1) = \sum_{n=0}^{\infty} P(X = 4n + 3) = \sum_{n=0}^{\infty} (1/2)^{4n+3} = \frac{1}{8} \sum_{n=0}^{\infty} (1/16)^n = \frac{1}{8} \frac{1}{1-1/16} = \frac{2}{15}
\]

And we note that \( P(Y = 1) + P(Y = 0) + P(Y = -1) = 1 \)

(c) We have

\[
P(X^2 + 2X > 8) = P(X^2 + 2X - 8 > 0) = P((X + 4)(X - 2) > 0)
\]

Now if \((X + 4)(X - 2) > 0\), then it must be the case that either

\[
(X + 4) > 0 \cap (X - 2) > 0 \quad \text{or} \quad (X + 4) < 0 \cap (X - 2) < 0
\]

However, the second case can never happen, since \( P(X < -4) = 0 \). so

\[
P(X^2 + 2X > 8) = P(X > -4 \cap X > 2) = P(X > 2) = 1 - P(X = 1) - P(X = 2) = 1/4
\]
(d) When \( m \leq 0 \), we have \( P(X \geq m) = 1 \), and for integer \( m \geq 1 \), we have

\[
P(X \geq m) = \sum_{k=m}^{\infty} P(X = k) = \sum_{k=1}^{m-1} (1/2)^k - \sum_{k=1}^{m-1} (1/2)^k = \frac{1/2}{1 - 1/2} = \frac{1/2 - (1/2)^m}{1 - 1/2} = (1/2)^{m-1}
\]

Now if \( P(X \geq m) = 1/2 \), then \((1/2)^{m-1} = 1/2\) which implies \( m = 2 \). We also note that since \( X \) only takes on the values 1, 2, 3, \ldots with non-zero probability, we have \( P(X \geq 2) = 1 - P(X = 1) = 1/2 \).

Problem 4.3

Suppose we know that a discrete random variable \( X \) has mean \( m \) and variance \( \sigma^2 \). We also know that \( X \) is uniform over a set of 2 integers.

(a) Find the PMF of \( X \).

(b) Now let \( Y = \frac{X - m}{\sigma} \). Find the PMF, the mean, and the variance of \( Y \).

(c) Find the variance of \( \cos(\pi Y) \).

(d) In general, if the PMF \( P(X = k) \) is an even function, then what is the expected value of \( X \)?

Solutions

(a) We know that there exist integers \( a > b \) such that

\[
P(X = k) = \begin{cases} 
\frac{1}{2} & \text{if } k = a, b \\
0 & \text{otherwise}
\end{cases}
\]

So

\[
m = E[X] = \frac{a + b}{2}
\]

and

\[
\sigma^2 = Var[X] = E[X^2] - E[X]^2 = \frac{a^2 + b^2}{2} - \frac{a^2 + b^2 + 2ab}{4} = \frac{a^2 + b^2 - 2ab}{4} = \frac{(a - b)^2}{4}.
\]

Together these imply

\[
a + b = 2m \quad \text{and} \quad a - b = 2\sigma
\]

and so solving for \( a \) and \( b \) yields

\[
a = m + \sigma \quad \text{and} \quad b = m - \sigma
\]

(b) We have

\[
P(Y = k) = P\left( \frac{X - m}{\sigma} = k \right) = P(X = \sigma k + m) = \begin{cases} 
\frac{1}{2} & \text{if } \sigma k + m = m \pm \sigma \\
0 & \text{otherwise}
\end{cases}
\]

So \( E[Y] = \frac{1}{2}(1 - 1) = 0 \), and \( Var[Y] = E[Y^2] = \frac{1}{2} \left( 1^2 + (-1)^2 \right) = 1 \).
Alternatively, if \( Z = aX + b \), for some constants \( a \) and \( b \), then by the linearity of expected value, we have
\[
E[Z] = E[aX + b] = aE[X] + b
\]

and
\[
Var[Z] = E[Z^2] - (E[Z])^2 = E[(aX + b)^2] - E[aX + b]^2
\]
\[
= (a^2E[X^2] + 2abE[X] + b^2) - (a^2E[X]^2 + 2abE[X] + b^2)
\]
\[
= a^2(E[X^2] - E[X]^2) = a^2Var[X]
\]

Since \( Y = \frac{X - m}{\sigma} \), if we take \( a = 1/\sigma \) and \( b = -m/\sigma \), we get
\[
E[Y] = \frac{1}{\sigma} E[X] - \frac{m}{\sigma} = \frac{m}{\sigma} - \frac{m}{\sigma} = 0
\]

and
\[
Var[Y] = \frac{1}{\sigma^2} Var[X] = 1.
\]

(c) We know that \( P(Y = 1) = P(Y = -1) = 1/2 \), so
\[
E[\cos(\pi Y)] = \frac{1}{2} \cos(\pi) + \frac{1}{2} \cos(-\pi) = -1
\]

and
\[
E[\cos^2(\pi Y)] = E\left[\frac{1 + \cos(2\pi Y)}{2}\right] = \frac{1}{2} + \frac{1}{2} E[\cos(2\pi Y)] = \frac{1}{2} + \frac{1}{2} (\cos(2\pi)/2 + \cos(-2\pi)/2) = 1
\]

Thus
\[
Var[\cos(\pi Y)] = 1 - (\frac{-1}{2})^2 = 0.
\]

\( Y \in \{-1, 1\} \), and \( \cos(\pi) = \cos(-\pi) \), so \( \cos(\pi Y) = 1 \) with probability 1.

(d) If \( P(X = k) \) is even, then \( P(X = k) = P(X = -k) \) for all \( k \). Thus
\[
E[X] = \sum_k k P(X = k) = 0 P(X = 0) + \sum_{k>0} k P(X = k) - k P(X = -k)
\]
\[
= \sum_{k>0} k (P(X = k) - P(X = k)) = 0.
\]
Problem 4.4

7 people each owns a hat, and we assign the 7 hats to the 7 people at random, i.e., all possibilities of who gets which hat are equally likely. Let $X$ be the number of people who get their own hat back. Find the expected value of $X$.

Solutions

In Problem 2.4, we calculated the probability that at least one person out of $N$ gets their own hat back. We denote this probability by:

$$q(N) = -\sum_{k=1}^{N} \frac{(-1)^k}{k!}$$

Let $k \in \{0, 1, 2, 3, 4, 5, 6, 7\}$, and let $A_1, \ldots, A_7$ denote the events that each person gets their own hat back. When $X = k$, exactly $k$ people get their hat back. By a symmetry argument, the probability that any particular group of $k$ people should be the same regardless of the group, so

$$P(X = k) = \binom{7}{k} (1-q(7-k)) \frac{(7-k)!}{7!}$$

There are $\binom{7}{k}$ ways of picking $k$ people from 7, and when $A_1 \cap \cdots \cap A_k$ occur, the first $k$ hats are fixed at the first $k$ people, and the remaining $(7-k)$ hats must end up with distinct people. So

$$P(A_1 \cap \cdots \cap A_k \cap A_{k+1}^{c} \cap \cdots \cap A_7^{c}) = P(A_{k+1}^{c} \cap \cdots \cap A_7^{c} \mid A_1 \cap \cdots \cap A_k) \cdot P(A_1 \cap \cdots \cap A_k)$$

We note that $P(A_1 \cap \cdots \cap A_k) = \frac{(7-k)!}{7!},$ since we have

$$P(A_1) = 1/7, \quad P(A_2 \mid A_1) = 1/6, \quad P(A_3 \mid A_1 \cap A_2) = 1/5, \quad P(A_4 \mid A_1 \cap A_2 \cap A_3) = 1/4, \cdots$$

Thus we have

$$P(X = k) = \binom{7}{k} (1-q(7-k)) \frac{(7-k)!}{7!} = \frac{1}{k!} \sum_{l=0}^{7-k} \frac{(-1)^l}{l!}$$

and in particular,

$$P(X = 0) = \frac{103}{280}, \quad P(X = 1) = \frac{53}{144}, \quad P(X = 2) = \frac{11}{60}, \quad P(X = 3) = \frac{1}{16},$$

$$P(X = 4) = \frac{1}{72}, \quad P(X = 5) = \frac{1}{240}, \quad P(X = 6) = 0, \quad P(X = 7) = \frac{1}{5040}$$

We note that $P(X = 0) + P(X = 1) + \cdots + P(X = 7) = 1$.

Also, $P(X = 6) = 0$ is reasonable, since if 6 of the 7 people have their own hats, the 7th person must also have their own hat.

Thus the expected number of correctly-returned hats is:

$$E[X] = 0 \frac{103}{280} + 1 \frac{53}{144} + 2 \frac{11}{60} + 3 \frac{1}{16} + 4 \frac{1}{72} + 5 \frac{1}{240} + 6 (0) + 7 \frac{1}{5040} = 1$$

Alternatively, let $X_i = 1$ when person $i$ gets their own hat back, and $X_i = 0$ otherwise. Then

$$P(X_i) = 1/7$$

since person $i$ is equally likely to get any of the 7 hats. Then the expected number of correctly-returned hats is:

$$E[X_1 + X_2 + \cdots + X_7] = E[X_1] + E[X_2] + \cdots + E[X_7] = 1$$
Problem 4.5

Suppose a $20 scratch-off ticket has the following odds

<table>
<thead>
<tr>
<th>Win amount</th>
<th>Odds</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20</td>
<td>1 in 4</td>
</tr>
<tr>
<td>$30</td>
<td>1 in 40</td>
</tr>
<tr>
<td>$50</td>
<td>1 in 40</td>
</tr>
<tr>
<td>$100</td>
<td>1 in 40</td>
</tr>
<tr>
<td>$250</td>
<td>1 in 800</td>
</tr>
<tr>
<td>$500</td>
<td>1 in 1,200</td>
</tr>
<tr>
<td>$1000</td>
<td>1 in 1,600</td>
</tr>
<tr>
<td>$10,000</td>
<td>1 in 40,000</td>
</tr>
<tr>
<td>$20,000,000</td>
<td>1 in 8,000,000 (Jackpot!)</td>
</tr>
</tbody>
</table>

(a) If we buy tickets and stop once we win at least $50 on a ticket, what is the expected number of tickets bought?

(b) If all other amounts and probabilities are fixed, what amount of money would the jackpot need to be in order for our expected winnings on a single ticket to exceed $20?

Solutions

(a) Any of the pay outs except $20 and $30 result in winning at least $50, so the probability $p$, we win at least $50 from a single ticket is

$$ p = \frac{1}{40} + \frac{1}{800} + \frac{1}{1,200} + \frac{1}{1,600} + \frac{1}{40,000} + \frac{1}{8,000,000} \approx 0.0527 $$

Recall

$$ \sum_{k=1}^{n} k a^k = a \frac{1 - (n + 1) a^n + n a^{n+1}}{(1-a)^2} $$

(This follows from taking a derivative of the geometric series)

If $|a| < 1$, then

$$ \sum_{k=1}^{\infty} k a^k = \lim_{n \to \infty} a \frac{1 - (n + 1) a^n + n a^{n+1}}{(1-a)^2} = \frac{a}{(1-a)^2}. $$

For each $n = 1, 2, \ldots$, if we buy $n$ tickets, then the first $n-1$ tickets won us less than $50 each, and the $n$th ticket won us at least $50. So

$$ P(\text{we buy } n \text{ tickets}) = (1 - p)^{n-1} p $$

$$ E[\text{number of tickets}] = \sum_{n=1}^{\infty} n P(\text{we buy } n \text{ tickets}) $$

$$ = \sum_{n=1}^{\infty} n (1 - p)^{n-1} p = \frac{p}{1 - p} \sum_{n=1}^{\infty} n (1 - p)^n $$

$$ = \frac{p}{1 - p} \frac{1 - p}{p^2} = \frac{1}{p} \approx 18.96 $$

We also note that this is a Geometric Random Variable with $p = 0.0527$. 

(b) Let $y$ be the payout of the jackpot.

$$E[X] = \frac{20}{40} + \frac{30}{40} + \frac{50}{800} + \frac{100}{1200} + \frac{250}{4000} + \frac{500}{8000} + \frac{1000}{16000} + \frac{10000}{240000} + \frac{y}{8000000} = \frac{3y + 266500000}{24000000}$$

If $E[X] \geq 20$, then

$$y \geq \frac{20(24,000,000) - 266,500,000}{3} \approx 71,166,667$$

We note that if $y = 20,000,000$, then $E[X] = 13.6$. Which means we expect to lose about $6.4$ with every ticket we buy (or if we are the lottery, we make an expected $6.4$ for every ticket sold).