Problem 1.1
Socks in a drawer are either long or short, either blue or red, and either cotton or wool. Of the 10 socks in the drawer, 5 of them are long and blue, 2 of them are long and red, and 2 of them red and short. Also, wool socks are neither blue nor short. What is the probability that a randomly selected sock is:

(a) long?
(b) blue?
(c) either long or red?

Solutions

(a) Let $B, C$, and $L$ be the events that a sock is blue, cotton, and long, respectively. We know:

\[ P(BL) = 0.5, \quad P(B^cL) = 0.2, \quad P(B^cL^c) = 0.2, \quad \text{and} \quad P(B^cC^c) = P(C^cL^c) = 0. \]

$L = (BL) \cup (B^cL)$ is a disjoint union of events, so

\[ P(L) = P(BL) + P(B^cL) = 0.5 + 0.2 = 0.7. \]

(b) We can express the sample space as a disjoint union of events:

\[ S = (BL) \cup (B^cL) \cup (BL^c) \cup (B^cL^c) \]

∴ $1 = P(BL) + P(B^cL) + P(BL^c) + P(B^cL^c)$.

Solving for $P(BL^c)$ yields

\[ P(BL^c) = 1 - 0.5 - 0.2 - 0.2 = 0.1. \]

$B = (BL) \cup (BL^c)$ is a disjoint union of events, so

\[ P(B) = P(BL) + P(BL^c) = 0.5 + 0.1 = 0.6. \]

(c) By Demorgan’s Law and the fact $P(E^c) = 1 - P(E)$, we have

\[ P(B^c \cup L) = P((BL)^c) = 1 - P(BL^c) = 1 - 0.1 = 0.9. \]

Alternatively,

\[ P(B^c \cup L) = P(B^c) + P(L) - P(B^cL) = (1 - 0.6) + 0.7 - 0.2 = 0.9 \]
Problem 1.2

Suppose we roll two normal 6-sided dice. Find the probability that

(a) Exactly one die shows a 3.
(b) At least one die shows a 3.
(c) At least one die shows a 3 and the sum of the dice is at most 5.
(d) At least one die shows a 3 or the sum of the dice is at most 5.

Solutions

The sample space of this experiment is all pairs \((x, y) \in \{1, 2, 3, 4, 5, 6\}\)^2

(a)

\[
P(\text{"Exactly one die shows a 3"}) = P(\text{"First die is 3 and second is not"}) + P(\text{"Second die is 3 and first is not"})
\]
\[
= \frac{5}{36} + \frac{5}{36} = \frac{5}{18}
\]

(b)

\[
P(\text{"At least one die shows a 3"}) = P(\text{"Exactly one die shows a 3"}) + P(\text{"Both die show a 3"})
\]
\[
= \frac{10}{36} + \frac{1}{36} = \frac{11}{36}
\]

Alternatively, let \(A\) and \(B\) denote the events that the first and second die show a 3, respectively. Then

\[
P(A \cup B) = P(A) + P(B) - P(AB) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}
\]

(c) If the sum of the dice is at most 5 AND at least one die shows a 3, then the dice values must be in the set

\[
\{(3, 1), (3, 2), (2, 3), (1, 3)\}
\]

so the probability of this event is

\[
\frac{4}{36} = \frac{1}{9}
\]

(d) Let \(A\) denote the event that at least one shows a 3, and let \(B\) denote the event that the sum of the dice is at most 5. Then by (b) and (c), we know \(P(A)\) and \(P(AB)\). In order to calculate \(P(A \cup B)\), let’s calculate \(P(B)\). (There are other ways to do this problem, but we already have 2 out of the 3 pieces of information to do it this way).

If the sum is at most 5, then the dice must take on one of the following pairs

\[
(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)
\]

So \(P(B) = \frac{10}{36}.\) Thus

\[
P(A \cup B) = P(A) + P(B) - P(AB) = \frac{11}{36} + \frac{10}{36} - \frac{4}{36} = \frac{17}{36}
\]

If we had instead wanted the exclusive or, i.e. exactly one of \(A\) and \(B\) occurs, then we could write this as

\[
P(AB^c \cup A^c B) = P(A \cup B) - P(AB) = \frac{17}{36} - \frac{4}{36} = \frac{13}{36}
\]
Problem 1.3

(a) For any events \( A \) and \( B \) in a sample space \( S \), show that \( P(AB) \geq 1 - P(A^c) - P(B^c) \).

(b) Consider two standard coin flips. If \( A_1 \) and \( B_1 \) are the events the first and second coin, respectively, are heads, show the inequality in (a) is not tight.

(c) Consider two standard coin flips. If \( A_2 \) is the event either coin is heads and \( B_2 \) is the event either coin is tails, show the inequality in (a) is tight.

(d) If the inequality is tight in (a), what is the relationship between sets \( A \) and \( B^c \)?

Solutions

(a) Using DeMorgan’s Law and the fact \( P(E^c) = 1 - P(E) \), we have
\[
P(AB) = P((A^c \cup B^c)^c) = 1 - P(A^c \cup B^c) = 1 - (P(A^c) + P(B^c) - P(A^cB^c))
\]
and \( P(A^cB^c) \geq 0 \), so
\[
P(AB) = 1 + P(A^cB^c) - P(A^c) - P(B^c) \geq 1 - P(A^c) - P(B^c).
\]

(b) For two standard coin flips, we have \( S = \{HH, HT, TH, TT\} \) and
\[
P(A_1^c) = P(\{TH, TT\}) = 1/2
\]
\[
P(B_1^c) = P(\{HT, TT\}) = 1/2
\]
\[
P(A_1B_1) = P(\{HH\}) = 1/4.
\]
Thus, \( P(A_1B_1) = 1/4 > 0 = 1 - P(A_1^c) - P(B_1^c) \).

(c) For two standard coin flips, we have \( S = \{HH, HT, TH, TT\} \) and
\[
P(A_2^c) = P(\{TT\}) = 1/4
\]
\[
P(B_2^c) = P(\{HH\}) = 1/4
\]
\[
P(A_2B_2) = P(\{HT, TH\}) = 1/2.
\]
Thus, \( P(A_2B_2) = 1/2 = 1 - P(A_2^c) - P(B_2^c) \).

(d) If \( P(AB) = 1 - P(A^c) - P(B^c) \), then \( P(A^cB^c) = 0 \). So
\[
P(B^c) = P(A^cB^c) + P(AB^c) = P(AB^c)
\]
which implies \( AB^c = B^c \). Therefore, \( B^c \subseteq A \).

In part (c), \( B_2^c = \{HH\} \subset \{HH, HT, TH\} = A_1 \).
Problem 1.4

Let $A$, $B$, and $C$ be events in the sample space $S$. Write the following probability in terms of a sum of disjoint events.

(a) $E_1$ is the event exactly one of the events occurs.

(b) $E_2$ is the event that at least one of $B$ and $C$ occurs but $A$ does not.

(c) $E_3$ is the event that $A$ occurs but $B$ does not.

(d) $E_4$ is the event that $C$ occurs and exactly one of $A$ and $B$ occur.

(e) Now suppose $P(E_2) = 0.3$, $P(E_3) = 0.2$, and $P(E_4) = 0.1$. Find $P(E_1)$.

Solutions

(a) Only $A$ occurs: $AB^cC^c$, only $B$ occurs: $A^cBC^c$, only $C$ occurs: $A^cB^cC$, so

$$P(E_1) = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC).$$

(b) At least one of $B$ and $C$ occurs but $A$ does not:

$$E_2 = A^c(B \cup C) = A^cBC^c \cup A^cBC \cup A^cB^cC^c,$$

so

$$P(E_2) = P(A^cBC) + P(A^cBC^c) + P(A^cB^cC).$$

(c) $A$ occurs but $B$ does not: $AB^c = AB^cC \cup AB^cC^c$, so

$$P(E_3) = P(AB^cC) + P(AB^cC^c).$$

(d) $C$ occurs and exactly one of $A$ and $B$ occur: $AB^cC \cup A^cBC$, so

$$P(E_4) = P(AB^cC) + P(A^cBC).$$

(e) We have

$$P(E_1) = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC)$$

and

$$P(E_2) + P(E_3) = \left[P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC)\right] + \left[P(A^cBC) + P(AB^cC)\right]$$

$$= P(E_1) + P(E_4)$$

Solving for $E_1$ yields

$$P(E_1) = 0.3 + 0.2 - 0.1 = 0.4.$$