Problem 9.1

Determine the following for random variables $X, Y$ given by

$$f_{X,Y}(u, v) = \begin{cases} \lambda^2 e^{-\lambda(u+v)} & \text{if } u, v \geq 0 \\ 0 & \text{else} \end{cases} \quad (\text{where } \lambda > 0)$$

(a) Are $X$ and $Y$ independent?

(b) For what value of $\lambda$ does $E[XY + 2Y + X - 4] = 0$?

(c) Find the probability $X > Y$.

Solutions

(a) $X$ and $Y$ are independent if and only if $f_{X,Y}(u, v) = f_X(u) f_Y(v)$.

We have

$$f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u, v) \, dv = \begin{cases} \int_0^{\infty} \lambda^2 e^{-\lambda(u+v)} \, dv & \text{if } u \geq 0 \\ 0 & \text{if } u < 0 \end{cases} = \begin{cases} \lambda e^{-\lambda u} & \text{if } u \geq 0 \\ 0 & \text{else} \end{cases}$$

By symmetry argument, $f_X(u) = f_Y(u)$.

It is easy to verify $X$ and $Y$ are independent. In fact, $X$ and $Y$ are I.I.D. (independent, identically distributed).

(b) By linearity of expected value, we have $E[XY + 2Y + X - 4] = E[XY] + 2 E[Y] + E[X] - 4$

$X$ and $Y$ are identically distributed, so $E[X] = E[Y]$, and $X$ and $Y$ are $\lambda$ exponential random variables, so $E[X] = E[Y] = \frac{1}{\lambda}$

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u v f_{X,Y}(u, v) \, du \, dv = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u v f_X(u) f_Y(v) \, du \, dv$$

$$= \left( \int_{-\infty}^{\infty} u f_X(u) \, du \right) \left( \int_{-\infty}^{\infty} v f_Y(v) \, dv \right) = E[X] E[Y] = \frac{1}{\lambda^2}$$

Note that $E[XY] = E[X] E[Y]$. This is NOT generally true, but when it is, $X$ and $Y$ are said to be uncorrelated. Note that independence implies uncorrelated, but not vice versa. i.e. there are random variables which are uncorrelated but not independent.

So we have

$$0 = E[XY + 2Y + X - 4] = \frac{1}{\lambda^2} + \frac{3}{\lambda} - 4 = \left( \frac{1}{\lambda} + 4 \right) \left( \frac{1}{\lambda} - 1 \right) \Rightarrow \lambda = 1 \text{ or } -\frac{1}{4}$$

$\lambda > 0$, so we have $E[XY + 2Y + X - 4] = 0$, when $\lambda = 1$
\( P(X > Y) = \int_{u > v} \int f_{X,Y}(u,v) \, du \, dv = \int_{0}^{\infty} \int_{0}^{\infty} \lambda^2 e^{-\lambda(u+v)} \, du \, dv = \int_{0}^{\infty} \lambda e^{-2\lambda v} \, dv = \frac{1}{2} \)

In fact, whenever the PDF of \( X \) and \( Y \) is symmetric, i.e. \( f_{X,Y}(u,v) = f_{X,Y}(v,u) \), we have

\( P(X > Y) = \int_{u > v} \int f_{X,Y}(u,v) \, du \, dv = \int_{u > v} \int f_{X,Y}(v,u) \, du \, dv = P(Y > X) \)

and \( 1 = P(X > Y) + P(Y > X) \), which implies \( P(X > Y) = P(Y > X) = 1/2 \).

Problem 9.2

Suppose \( X \) is a continuous random variable with PDF \( f_X(u) \). Find the value of \( a \) that minimizes \( E[(X - a)^2] \). What is \( E[(X - a)^2] \) in this case?

Comment: This arises as a way of measuring the error of estimating a value of \( X \). Taking \((x - y)^2\) is a way of measuring the “difference” between \( x \) and \( y \). In particular, if one wants to quantize \( X \) to a single point \( a \), what is the “best” estimate of \( X \)?

Solutions

We have

\[
E[(X - a)^2] = \int (u - a)^2 f_X(u) \, du = \int u^2 f_X(u) \, du - 2a \int u f_X(u) \, du + a^2 \int f_X(u) \, du
\]

\[
= E[X^2] - 2aE[X] + a^2
\]

If \( a \) minimizes \( E[(X - a)^2] \), then

\[
0 = \frac{d}{da} E[(X - a)^2] = 2a - 2E[X]
\]

and

\[
\frac{d^2}{da^2} E[(X - a)^2] = 2 > 0
\]

which implies \( E[(X - a)^2] \) is minimized by taking \( a = E[X] \), and in this case, \( E[(X - a)^2] = Var[X] \).
Problem 9.3

Let $X$ be uniform on the interval $[-a, a]$, for some $a > 0$, and let $Y = X^2$.

(a) Find the Pearson correlation coefficient between $X$ and $Y$.

(b) Are $X$ and $Y$ independent?

Solutions

(a) Let $n$ be a positive integer, then

$$
E[X^n] = \int_{-a}^{a} u^n \frac{1}{2a} \, du = \frac{a^{n+1} - (-a)^{n+1}}{2a(n+1)} = \begin{cases} 
0 & \text{if } n \text{ is odd} \\
\frac{a^n}{(n+1)} & \text{if } n \text{ is even}.
\end{cases}
$$

$$
\text{Var}[X] = E[(X - E[X])^2] = E[X^2] - E[X]^2 = \frac{a^2}{3}
$$

$$
$$

$$
$$

which implies

$$
\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\text{Var}[Y]}} = 0
$$

so $X$ and $Y$ are uncorrelated.

(b) Intuitively, $X$ and $Y$ are very dependent on one another, since if we know $X$, we know $Y$ is equal to $X^2$, and if we know $Y$, we know $X$ is equal to $\pm \sqrt{Y}$. Formally, if we show there exists a pair $(x, y) \in \mathbb{R}$ such that

$$
P(X \leq x, Y \leq y) \neq P(X \leq x)P(Y \leq y)
$$

then $X$ and $Y$ are not independent.

Let’s take $x = a/2$ and $y = a^2/4$. Then

$$
P(X \leq a/2, Y \leq a/4) = P(X \leq a/2, X^2 \leq a^2/4) = P(X \leq a/2, -a/2 \leq X \leq a/2) = P(-a/2 \leq X \leq a/2) = P(X^2 \leq a^2/4) = P(Y \leq a^2/4)
$$

but $P(X \leq a/2) \neq 1$, so $X$ and $Y$ are not independent.

Therefore random variables being uncorrelated does not imply they are independent.
Problem 9.4

Suppose $A$ and $B$ are independent random variables such that

$$P(A = 0) = P(A = 1) = P(B = 0) = P(B = 1) = 1/2$$

and let $X = A + B$ and $Y = A - B$.

(a) Are $X$ and $Y$ independent? Find the joint PMF of $X$ and $Y$

(b) Are $X$ and $Y$ uncorrelated?

Solutions

(a) We have

<table>
<thead>
<tr>
<th>$(A, B)$</th>
<th>$(A + B, A - B)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>(1, -1)</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>(2, 0)</td>
</tr>
</tbody>
</table>

which implies

$$P(X = 0, Y = 0) = P(X = 1, Y = -1) = P(X = 1, Y = 1) = P(X = 2, Y = 0) = 1/4$$

and $P(X = x, Y = y) = 0$, otherwise. However, $P(X = 2) = P(A = 1, B = 1) = 1/4$ and $P(Y = 1) = P(A = 1, B = 0) = 1/4$, so

$$P(X = 2, Y = 1) = 0 \neq P(X = 2)P(Y = 1).$$

Thus $X$ and $Y$ are not independent

(b) We have

$$E[X] = 0P(X = 0, Y = 0) + (P(X = 1, Y = -1) + P(X = 1, Y = 1) + 2P(X = 2, Y = 0) = 1$$

$$E[Y] = -P(X = 1, Y = 1) + 0(P(X = 0, Y = 0) + P(X = 2, Y = 0)) + P(X = 1, Y = 1) = 0$$

$$E[XY] = 0(P(X = 0, Y = 0) + P(X = 2, Y = 0)) - P(X = 1, Y = -1) + P(X = 1, Y = 1) = 0$$

which implies $Cov[X, Y] = E[XY] - E[X]E[Y] = 0$. Therefore $X$ and $Y$ are uncorrelated.