Problem 6.1

Suppose a $20 scratch-off ticket has the following odds

<table>
<thead>
<tr>
<th>Win amount</th>
<th>Odds</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20</td>
<td>1 in 4</td>
</tr>
<tr>
<td>$30</td>
<td>1 in 40</td>
</tr>
<tr>
<td>$50</td>
<td>1 in 40</td>
</tr>
<tr>
<td>$100</td>
<td>1 in 40</td>
</tr>
<tr>
<td>$250</td>
<td>1 in 800</td>
</tr>
<tr>
<td>$500</td>
<td>1 in 1200</td>
</tr>
<tr>
<td>$1000</td>
<td>1 in 1600</td>
</tr>
<tr>
<td>$10000</td>
<td>1 in 40000</td>
</tr>
<tr>
<td>$200000000</td>
<td>1 in 8000000 (Jackpot!)</td>
</tr>
</tbody>
</table>

(a) If we buy tickets and stop once we win at least $20 on a ticket, what is the expected number of tickets bought?

(b) If all other amounts and probabilities are fixed, what amount of money would the jackpot need to be in order for our expected winnings on a single ticket to exceed $20?

Solutions

(a) Any of the payouts result in winning at least $20, so the probability, we win at least $20 from a single ticket is

\[
\frac{1}{4} + \frac{1}{40} + \frac{1}{40} + \frac{1}{800} + \frac{1}{1200} + \frac{1}{1600} + \frac{1}{40000} + \frac{1}{8000000} \approx 0.3277
\]

Recall \( \sum_{k=1}^{n} k a^k = a \frac{1-(n+1)a^n + na^{n+1}}{(1-a)^2} \).

If \( |a| < 1 \), then \( \sum_{k=1}^{\infty} k a^k = \lim_{n \to \infty} a \frac{1-(n+1)a^n + na^{n+1}}{(1-a)^2} = \frac{a}{(1-a)^2} \).

For each \( n = 1, 2, \ldots \), if we buy \( n \) tickets, then the first \( n-1 \) tickets won us no money and the \( n \)th ticket won us at least $20. So

\[
P(\text{we buy } n \text{ tickets}) = (1 - 0.3277)^{n-1} 0.3277 = (0.6723)^{n-1} 0.3277
\]

\[
E[\text{number of tickets}] = \sum_{n=1}^{\infty} n P(\text{we buy } n \text{ tickets}) = \sum_{n=1}^{\infty} n (0.6723)^{n-1} 0.3277 = 0.3277 \sum_{n=1}^{\infty} n (0.6723)^n
\]

\[
= 0.3277 \frac{0.6723}{0.6723 (1 - 0.6723)^2} = \frac{1}{0.3277} \approx 3.05
\]

We also note that this is simply a Geometric Random Variable with \( p = 0.3277 \).

Please report any typos/errors to j2conelly@uscd.edu
(b) Let \( y \) be the payout of the jackpot. 

\[
E[X] = \frac{20}{4} + \frac{30}{40} + \frac{50}{40} + \frac{100}{800} + \frac{250}{1200} + \frac{500}{1600} + \frac{1000}{40000} + \frac{y}{8000000} = \frac{3y + 266500000}{24000000}
\]

If \( E[X] \geq 20 \), then

\[
y \geq \frac{20(24000000) - 266500000}{3} \approx 7116667
\]

We note that if \( y = 20000000 \), then \( E[X] = 13.6 \). Which means we expect to lose about $6.4 with every ticket we buy.

**Problem 6.2**

Suppose we are enrolled in two classes. In each class, the probability we get an A is \( \frac{1}{6} \), the probability we get a B is \( \frac{1}{2} \), and the probability we get a C is \( \frac{1}{3} \). Both classes are the same number of credits, our grade in one class is independent of our grade in the other, and \( A = 4.0 \), \( B = 3.0 \), and \( C = 2.0 \) points. Our GPA is simply the average of the points.

(a) Find the probability that our GPA is at least 3.4.

(b) Find our expected GPA.

(c) Find the probability we got at least one B given that our GPA is between 2.4 and 3.2.

**Solutions**

(a) Let \( A_i, B_i, C_i \) be the events we get an A, B, and C, respectively, in the \( i \)th class. Then

<table>
<thead>
<tr>
<th>Class 1</th>
<th>Class 2</th>
<th>GPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>A</td>
<td>4.0</td>
</tr>
<tr>
<td>A</td>
<td>B</td>
<td>3.5</td>
</tr>
<tr>
<td>A</td>
<td>C</td>
<td>3.0</td>
</tr>
<tr>
<td>B</td>
<td>A</td>
<td>3.5</td>
</tr>
<tr>
<td>B</td>
<td>B</td>
<td>3.0</td>
</tr>
<tr>
<td>B</td>
<td>C</td>
<td>2.5</td>
</tr>
<tr>
<td>C</td>
<td>A</td>
<td>3.0</td>
</tr>
<tr>
<td>C</td>
<td>B</td>
<td>2.5</td>
</tr>
<tr>
<td>C</td>
<td>C</td>
<td>2.0</td>
</tr>
</tbody>
</table>

\[
P(GPA = 4.0) = P(A_1A_2) = \frac{1}{36}
\]

\[
P(GPA = 3.5) = P(A_1B_2) + P(B_1A_2) = \frac{6}{36}
\]

\[
P(GPA = 3.0) = P(A_1C_2) + P(C_1A_2) + P(B_1B_2) = \frac{13}{36}
\]

\[
P(GPA = 2.5) = P(B_1C_2) + P(C_1B_2) = \frac{12}{36}
\]

\[
P(GPA = 2.0) = P(C_1C_2) = \frac{4}{36}
\]

Then \( P(GPA \geq 3.4) = P(GPA = 3.5) + P(GPA = 4.0) = \frac{7}{36} \approx 0.194 \).

(b) 

\[
E[GPA] = (4.0)\frac{1}{36} + (3.5)\frac{6}{36} + (3.0)\frac{13}{36} + (2.5)\frac{12}{36} + (2.0)\frac{4}{36} = \frac{17}{6} \approx 2.83
\]

(c) 

\[
P(B_1 \cup B_2 \mid 2.4 \leq GPA \leq 3.2) = \frac{P([B_1 \cup B_2] \cap [2.4 \leq GPA \leq 3.2])}{P(2.4 \leq GPA \leq 3.2)} = \frac{P(B_1B_2) + P(B_1C_2) + P(C_1B_2)}{P(GPA = 2.5) + P(GPA = 3.0)} = \frac{1/4 + 1/6 + 1/6}{12/36 + 13/36} = \frac{21}{25} = 0.84
\]
Problem 6.3

For random variable $X$ with probability density function

$$f_X(u) = e^{-2|u|}$$

(a) Verify $f_X(u)$ is a valid PDF and find $P(X^2 < 4 - 3X)$

(b) Find the expected value of $X$

(c) In general, if the PDF $f_X(u)$ of a random variable $X$ is even, what is the expected value of $X$?

Solutions

(a) Note that $f_X(u) \geq 0$ for all $u$, and we can write

$$f_X(u) = \begin{cases} e^{2u} & u < 0 \\ e^{-2u} & u \geq 0 \end{cases}$$

$$P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f_X(u) \, du = \int_{-\infty}^{0} e^{2u} \, du + \int_{0}^{\infty} e^{-2u} \, du = \frac{1}{2} + \frac{1}{2} = 1$$

Thus $f_X(u)$ is a valid density function.

$$P(X^2 < 4 - 3X) = P(X^2 + 3X - 4 < 0) = P([X + 4][X - 1] < 0)$$

$$= P(X + 4 < 0 \cap X - 1 > 0) + P(X + 4 > 0 \cap X - 1 < 0)$$

$$= P(X > -4 \cap X < 1) + P(X < -4 \cap X > 1)$$

$$= P(-4 < X < 1) = \int_{-4}^{1} f_X(u) \, du$$

$$= \int_{-4}^{0} e^{2u} \, du + \int_{0}^{1} e^{-2u} \, du = \frac{2 - e^{-8} - e^{-2}}{2}$$

(b)

$$E[X] = \int_{-\infty}^{\infty} u \, f_X(u) \, du = \int_{-\infty}^{0} u e^{2u} \, du + \int_{0}^{\infty} u e^{-2u} \, du = 0$$

(c) $f_X(u)$ is even, so $f_X(u) = f_X(-u)$. So

$$E[X] = \int_{-\infty}^{0} u \, f_X(u) \, du + \int_{0}^{\infty} u \, f_X(u) \, du = \int_{0}^{\infty} -v \, f_X(-v) \, (-dv) + \int_{0}^{\infty} u \, f_X(u) \, du$$

$$= -\int_{0}^{\infty} v \, f_X(-v) \, dv + \int_{0}^{\infty} u \, f_X(u) \, du = -\int_{0}^{\infty} v \, f_X(v) \, dv + \int_{0}^{\infty} u \, f_X(u) \, du = 0.$$