Problem 5.1

The arrival time of an employee can be modeled as a continuous random variable $H$, with probability density function given by

$$f_H(t) = \begin{cases} \frac{e^t - 1}{e - 1} & 0 \leq t \leq 1 \\ 0 & \text{else} \end{cases}$$

where $t$ is the number of hours after 8 am.

(a) Verify $f_H(t)$ is a valid density function and find the CDF of $H$.

(b) Find the probability the employee will arrive by 8:50 given they have not arrived by 8:30

Solutions

(a) $f_H(t) \geq 0$, for all $t$.

$$\int_{-\infty}^{\infty} f_H(t) \, dt = \frac{e}{e - 1} \int_0^1 e^{-t} \, dt = \frac{e}{e - 1} - (e^{-1} - 1) = \frac{e - 1}{e - 1} = 1$$

So $f_H(t)$ is a valid density function.

$$F_H(t) = P[H \leq t] = \int_{-\infty}^t f_H(\tau) \, d\tau = \begin{cases} 0 & t < 0 \\ \frac{e}{e - 1} \int_0^t e^{-\tau} \, d\tau & 0 \leq t \leq 1 \\ 1 & t > 1 \end{cases}$$

We have $F_H(t) \geq 0$ for all $t$, $F_H(t)$ is continuous (so it is right-continuous), $\lim_{t \to \infty} F_H(t) = 1$, and $\lim_{t \to -\infty} F_H(t) = 0$. Thus $F_H(t)$ is a valid CDF.

(b) Let $A$ be the event the employee arrives by 8:50 and $B$ be the event they have not arrived by 8:30

$$P[B] = P[H > 1/2] = 1 - P[H \leq 1/2] = \frac{e^{1/2} - 1}{e - 1}$$

$$P[AB] = P[1/2 < H \leq 5/6] = P[H \leq 5/6] - P[H \leq 1/2] = \frac{e^{1/2} - e^{5/6}}{e - 1}$$

$$P[A|B] = \frac{P[AB]}{P[B]} = \frac{e^{1/2} - e^{5/6}}{e^{1/2} - 1} \approx 0.72$$
Problem 5.2

Let $X$ be a continuous random variable. Suppose we flip a coin. If the coin is heads, then the PDF of $X$ is $f_X(u)$, and if the coin is tails, then the PDF of $X$ is $g_X(u)$, where

$$f_X(u) = \begin{cases} 
\frac{1}{2} & u \in [0, 2] \\
0 & \text{else} 
\end{cases} \quad \text{and} \quad g_X(u) = \begin{cases} 
\frac{1}{3} & u \in [1, 4] \\
0 & \text{else} 
\end{cases}.$$

(a) Find the probability the coin is heads, given $X \in [1, 3]$.

(b) Find the CDF and the PDF of $X$.

Solutions

(a) Let $H$ be the event the coin is heads. The PDF of $X$ depends on the event $H$, so

$$P(H \mid 1 \leq X \leq 3) = \frac{P(H \cap [1 \leq X \leq 3])}{P(1 \leq X \leq 3)} = \frac{P(1 \leq X \leq 3 \mid H)P(H)}{P(1 \leq X \leq 3 \mid H)P(H) + P(1 \leq X \leq 3 \mid H^c)P(H^c)} = \frac{\frac{1}{2} \int_1^3 f_X(u) \, du}{\frac{1}{2} \int_1^3 f_X(u) \, du + \frac{1}{2} \int_1^3 g_X(u) \, du} = \frac{\int_1^2 1/2 \, du}{\int_1^2 1/2 \, du + \int_1^3 1/3 \, du} = \frac{1/2}{1/2 + 2/3} = \frac{3}{7}.$$

(b)

$$F_X(u) = P(X \leq u) = P(X \leq u \cap H) + P(X \leq u \cap H^c) = P(H) P(X \leq u \mid H) + P(H^c) P(X \leq u \mid H^c) = \frac{1}{2} \int_{-\infty}^u f_X(u) \, du + \frac{1}{2} \int_{-\infty}^u g_X(u) \, du$$

$$\int_{-\infty}^u f_X(u) \, du = \begin{cases} 
0 & u < 0 \\
\int_0^u 1/2 \, du & 0 \leq u < 2 \\
\int_0^u 1/2 \, du & u \geq 2 
\end{cases} = \begin{cases} 
0 & u < 0 \\
u/2 & 0 \leq u < 2 \\
1 & u \geq 2 
\end{cases}$$

$$\int_{-\infty}^u g_X(u) \, du = \begin{cases} 
0 & u < 1 \\
\int_0^u 1/3 \, du & 1 \leq u < 4 \\
\int_1^u 1/2 \, du & u \geq 4 
\end{cases} = \begin{cases} 
0 & u < 1 \\
u/3 & 1 \leq u < 4 \\
1 & u \geq 4 
\end{cases}$$

Thus

$$F_X(u) = \begin{cases} 
0 & u < 0 \\
\frac{u}{4} & 0 \leq u < 1 \\
\frac{5u}{12} & 1 \leq u < 2 \\
\frac{u+3}{6} & 2 \leq u < 4 \\
1 & u \geq 4 
\end{cases} \quad \rightarrow \quad f_X(u) = \frac{d}{du} F_X(u) = \begin{cases} 
\frac{1}{4} & 0 \leq u < 1 \\
\frac{5}{12} & 1 \leq u < 2 \\
\frac{1}{6} & 2 \leq u < 4 \\
0 & \text{otherwise} 
\end{cases}.$$
Problem 5.3

A random variable \( X \) has distribution given by

\[
f_X(u) = \begin{cases} 
  u/2 & 0 \leq u \leq 2 \\
  0 & \text{else}
\end{cases}
\]

\( X \) is then used to create a new random variable \( Z \), where

\[
Z = \begin{cases} 
  X + 1 & X \leq 1 \\
  X - 1 & X > 1
\end{cases}
\]

(a) Find the CDF of \( X \)

(b) Find the probability \( Z \in [1/2, 3/2] \).

(c) Find the probability \( X > 1 \) given \( Z \geq 1 \)

Solutions

(a)

\[
F_X(u) = P[X \leq u] = \int_{-\infty}^{u} f_X(v) \, dv = \begin{cases} 
  0 & u < 0 \\
  \int_{0}^{u} v/2 \, dv & 0 \leq u \leq 2 \\
  1 & u > 2
\end{cases} = \begin{cases} 
  0 & u < 0 \\
  u^2/4 & 0 \leq u \leq 2 \\
  1 & u > 2
\end{cases}
\]

(b) \( Z \) is also a continuous random variable; however, \( Z \) is completely deterministic, once we know \( X \).

\[
P[1/2 \leq Z \leq 3/2] = P[(1/2 \leq Z \leq 3/2) \cap (X \leq 1)] + P[(1/2 \leq Z \leq 3/2) \cap (X > 1)] \\
= P[(1/2 \leq X + 1 \leq 3/2) \cap (X \leq 1)] + P[(1/2 \leq X - 1 \leq 3/2) \cap (X > 1)] \\
= P[(-1/2 \leq X \leq 1/2) \cap (X \leq 1)] + P[(3/2 \leq X \leq 5/2) \cap (X > 1)] \\
= P[0 \leq X \leq 1/2] + P[3/2 \leq X \leq 2] \\
= (F_X(1/2) - F_X(0)) + (F_X(2) - F_X(3/2)) \\
= (1/2)^2/4 - 0 + 1 - (3/2)^2/4 = \frac{1}{2}
\]

(c)

\[
P[X > 1 \mid Z \geq 1] = \frac{P[X > 1 \cap Z \geq 1]}{P[Z \geq 1]} = \frac{P[X > 1 \cap X - 1 \geq 1]}{P[Z \geq 1]} \\
= \frac{P[X > 1 \cap X \geq 2]}{P[Z \geq 1]} = 0
\]

where the last equality comes from the fact \( P[X \geq 2] = 0 \).