Problem 4.1
Suppose we flip a coin. If the coin is heads, we randomly pick an integer from \{-2, -1, 0, 1, 2\}. If the coin is tails, we randomly pick an integer from \{0, 1, 2, 3, 4\}. Let \(X\) be the integer that we pick.

(a) What is the PMF of \(X\)?

(b) What is the expected value of \(X\)?

(c) What is the probability the coin was heads, assuming the number we picked was 2?

(d) Suppose \(Y = X^2\). What is the PMF of \(Y\)?

Solutions

(a) Let \(H\) be the event the coin was heads, then

\[
P(X = k \mid H) = \begin{cases} 
\frac{1}{5} & \text{if } k = -2, -1, 0, 1, 2 \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
P(X = k \mid H^c) = \begin{cases} 
\frac{1}{5} & \text{if } k = 0, 1, 2, 3, 4 \\
0 & \text{otherwise}
\end{cases}
\]

so

\[
P(X = k) = P(X = k \mid H) P(H) + P(X = k \mid H^c) P(H^c) = \begin{cases} 
\frac{1}{10} & \text{if } k = -2, -1 \\
\frac{1}{5} & \text{if } k = 0, 1, 2 \\
\frac{1}{10} & \text{if } k = 3, 4 \\
0 & \text{otherwise}
\end{cases}
\]

To verify that this is valid, we note that \(\sum_{k=-2}^{4} P(X = k) = 1\).

(b) We have

\[
E[X] = \sum_{k=-2}^{4} k P(X = k) = \frac{-2}{10} + \frac{-1}{10} + \frac{1}{5} + \frac{2}{5} + \frac{3}{10} + \frac{4}{10} = 1
\]

(c)

\[
P(H \mid X = 2) = \frac{P(X = 2 \mid H) P(H)}{P(X = 2)} = \frac{(1/5)(1/2)}{(1/5)} = \frac{1}{2}
\]

which implies \(H\) and \(\{X = 2\}\) are independent events. However, if we know that \(X = 3\), then it must be the case that the coin was tails, so \(P(H \mid X = 3) = 0\), so the random variable \(X\) and the coin are clearly not independent in general.

(d) Clearly \(P(Y = k) = 0\) when \(k < 0\), since \(Y = X^2\). For \(k \geq 0\), we have

\[
P(Y = k) = P(X^2 = k) = P(X = \sqrt{k}) + P(X = -\sqrt{k}) = \begin{cases} 
\frac{1}{5} & \text{if } k = 0 \\
\frac{3}{10} & \text{if } k = 1, 4 \\
\frac{1}{10} & \text{if } k = 9, 16
\end{cases}
\]

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Problem 4.2

Suppose the PMF of a random variable \(X\) is

\[
P(X = k) = \begin{cases} 
    c^k & \text{if } k = 1, 2, \ldots \\
    0 & \text{otherwise.}
\end{cases}
\]

(a) What value should \(c\) take on?

(b) Let \(Y = \sin(X \pi / 2)\). What is the PMF of \(Y\)?

(c) What is the probability that \(P(X^2 + 2X > 8)\)?

(d) For what value of \(m\) do we have \(P(X \geq m) = 1/2\)?

Solutions

(a) We have

\[
1 = \sum_{k} P(X = k) = \sum_{k=1}^{\infty} c^k = \frac{c}{1 - c}
\]

which implies \(c = 1/2\).

(b) We have \(X \in \{1, 2, 3, 4, 5, 6, \ldots \}\).

When \(X = 1, 5, 9, 13, \ldots, 4n + 1, \ldots\), we have \(Y = \sin(\pi / 2) = 1\), so

\[
P(Y = 1) = \sum_{n=0}^{\infty} P(X = 4n + 1) = \sum_{n=0}^{\infty} (1/2)^{4n+1} = \frac{1}{2} \sum_{n=0}^{\infty} (1/16)^n = \frac{1}{2} \frac{1}{1 - 1/16} = \frac{8}{15}
\]

When \(X = 2, 6, 8, 10, \ldots, 2n, \ldots\), we have \(Y = \sin(\pi) = 0\), so

\[
P(Y = 0) = \sum_{n=1}^{\infty} P(X = 2n) = \sum_{n=1}^{\infty} (1/2)^{2n} = \sum_{n=1}^{\infty} (1/4)^n = \frac{1/4}{1 - 1/4} = \frac{1}{3}
\]

When \(X = 3, 7, 11, 15, \ldots, 4n + 3, \ldots\), we have \(Y = \sin(3\pi / 2) = -1\), so

\[
P(Y = -1) = \sum_{n=0}^{\infty} P(X = 4n + 3) = \sum_{n=0}^{\infty} (1/2)^{4n+3} = \frac{1}{8} \sum_{n=0}^{\infty} (1/16)^n = \frac{1}{8} \frac{1}{1 - 1/16} = \frac{2}{15}
\]

And we note that \(P(Y = 1) + P(Y = 0) + P(Y = -1) = 1\)

(c) We have

\[
P(X^2 + 2X > 8) = P(X^2 + 2X - 8 > 0) = P(X + 4)(X - 2) > 0
\]

Now if \((X + 4)(X - 2) > 0\), then it must be the case that either

\[(X + 4) > 0 \cap (X - 2) > 0 \quad \text{or} \quad (X + 4) < 0 \cap (X - 2) < 0\]

However, the second case can never happen, since \(P(X < -4) = 0\). so

\[
P(X^2 + 2X > 8) = P(X > -4 \cap X > 2) = P(X > 2) = 1 - P(X = 1) - P(X = 2) = 1/4
\]
(d) When \( m \leq 0 \), we have \( P(X \geq m) = 1 \), and for integer \( m \geq 1 \), we have

\[
P(X \geq m) = \sum_{k=m}^{\infty} P(X = k) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k - \sum_{k=1}^{m-1} \left(\frac{1}{2}\right)^k = \frac{1}{2 - \frac{1}{2}} - \frac{1/2 - (1/2)^m}{1 - \frac{1}{2}} = (1/2)^{m-1}
\]

Now if \( P(X \geq m) = 1/2 \), then \((1/2)^{m-1} = 1/2\) which implies \( m = 2 \). We also note that since \( X \) only takes on the values 1, 2, 3, \ldots \ with non-zero probability, we have

\[
P(X \geq 2) = 1 - P(X = 1) = 1/2.
\]