ECE 109 Discussion 2 Notes

Problem 2.1

From a deck of cards, we have the following cards:
2, 3, and 6 of hearts, 5, 8, 9, and 10 of clubs, 4, 6, 8, 9, and 10 of diamonds.
From this set of cards, we randomly draw a hand of 5 cards (the ordering in a hand does not matter).
What is the probability that there is a card from each of the 3 suits in our hand?

Solutions

Let $A$ be the event there is a card from each suit in our hand. Since selecting each card is equally likely,
we need only to count the number of ways $A$ could occur and divide by the number of possible hands.
We are selecting 5 cards from a set of 12 cards, so there are $\binom{12}{5} = 792$ hands we could draw.
We can count the number of ways $A$ does not occur and subtract that from the number of possible hands
to determine the number of ways $A$ occurs.

There are $\binom{12-3}{5} = 126$ ways of choosing 5 cards such that there are no hearts.
There are $\binom{12-4}{5} = 56$ ways of choosing 5 cards such that there are no clubs.
There are $\binom{12-5}{5} = 21$ ways of choosing 5 cards such that there are no diamonds.

However, we over counted situations where we are missing more than one suit. This is a similar idea to
the inclusion-exclusion principle, i.e.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$ 

There are $\binom{12-3-5}{5} = 0$ ways of choosing 5 cards such that there are no hearts or diamonds.
There are $\binom{12-4-5}{5} = 0$ ways of choosing 5 cards such that there are no clubs or diamonds.
There is $\binom{12-3-4}{5} = 1$ way of choosing 5 cards such that there are no hearts or clubs.
There are $\binom{12-12}{5} = 0$ ways of choosing 5 cards such that there are no hearts, clubs, or diamonds.

So there are $792 - (126 + 56 + 21 - 0 - 0 - 1 + 0) = 590$ ways $A$ could occur, so

$$P(A) = \frac{590}{792} = \frac{295}{396}$$
Problem 2.2

Suppose randomly we select 3 distinct numbers from the set \( \{1, 2, \ldots, 10\} \). What is the probability that at least one of the following occurs: we pick a 1, we pick a 2, or we pick a 3?

Solutions

There are \( \binom{10}{3} = 120 \) subsets of 3 distinct numbers from the set \( \{1, 2, \ldots, 10\} \) (disregarding order). Let \( A, B, \) and \( C \) denote the events we pick a 1, 2, and 3, respectively.

We have

\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)
\]

When one of the numbers is fixed, we are picking 2 distinct numbers from a set of 9 numbers, so

\[
P(A) = P(B) = P(C) = \frac{\binom{9}{2}}{\binom{10}{3}} = \frac{36}{120}
\]

When two of the numbers are fixed, we are picking 1 number from a set of 8 numbers so

\[
P(AB) = P(AC) = P(BC) = \frac{\binom{8}{1}}{\binom{10}{3}} = \frac{8}{120}
\]

When three of the numbers are fixed, there is only one possible way this could have occurred, so

\[
P(ABC) = \frac{\binom{10}{3}}{\binom{10}{3}} = \frac{1}{120}
\]

Therefore

\[
P(A \cup B \cup C) = \frac{3(36) - 3(8) + 1}{120} = \frac{85}{120} = \frac{17}{24} \approx 0.71
\]

Alternatively, there are \( \binom{10}{3}(9)(8) \) ways of choosing 3 distinct numbers from the set \( \{1, 2, \ldots, 10\} \), and there are \( \binom{7}{3}(6)(5) \) ways of choosing 3 distinct numbers from the set \( \{4, 5, \ldots, 10\} \) (accounting for order), so

\[
P(A \cup B \cup C) = 1 - P(\text{sequence does not contain 1,2, or 3}) = 1 - \frac{\binom{7}{3}(6)(5)}{\binom{10}{3}(9)(8)} = 1 - \frac{7}{24} = \frac{17}{24}
\]

which was much easier to calculate.
Problem 2.3

We randomly select a four-digit number. The number we select has distinct digits and does not contain a zero. What is the probability our number has a sum of digits equal to 10?

Solutions

Let $A$ be the event the number has a sum of digits equal to 10.

Let $B$ be the event the number has distinct digits and does not contain a zero.

There are \( \binom{9}{4} \) ways of choosing four distinct non-zero digits; however, the digits are ordered, so there are $4! \binom{9}{4} = \frac{9!}{5!}$ four-digit numbers with distinct non-zero digits. Alternatively, there are 9 choices of the first digit, 8 for the second, 7 for the third, and 6 for the fourth.

Since each four-digit number is equally likely,

$$P(B) = \frac{\text{number of outcomes satisfying } B}{\text{number of outcomes}} = \frac{(9)(8)(7)(6)}{10^4}.$$

If the digits are distinct, non-zero, and sum to 10, the digits must be 1, 2, 3, and 4. Thus,

$$P(AB) = \frac{\text{number of outcomes satisfying } A \text{ and } B}{\text{number of outcomes}} = \frac{4!}{10^4}.$$

Finally,

$$P(A|B) = \frac{P(AB)}{P(B)} = \frac{4!}{(9)(8)(7)(6)} = \frac{1}{126}.$$

Problem 2.4

$N$ people each owns a hat, and we randomly assign the $N$ hats to the $N$ people, i.e., all possibilities of who gets which hat are equally likely. Find the probability at least one person gets their hat back.

Let $A_1, \ldots, A_n$ denote the events that each person receives their own hat. Then the event that someone gets their own hat is $A_1 \cup \cdots \cup A_n$. So by the inclusion-exclusion principle, we have

$$P(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} P(A_1 \cap \cdots \cap A_k)$$

When $A_1 \cap \cdots \cap A_k$ occurs, the first $k$ people receive their own hats, but the remaining $n - k$ people can receive any of the remaining $(n - k)$ hats. So

$$P(A_1 \cap \cdots \cap A_k) = \frac{(n - k)!}{n!}$$

which implies

$$P(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \frac{(n - k)!}{n!}$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!}$$

$$= 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{24} + \cdots - \frac{(-1)^n}{n!}.$$