Problem 1.1

Socks in a drawer are either long or short, either black or white, and either cotton or wool. Given the probability a sock is long and black is 0.5, long and white is 0.2, white and short is 0.2, and wool socks are neither white nor short, what is the probability:

(a) a sock is long?

(b) a sock is black?

(c) a sock is long or white?

Solutions

(a) Let \( B, C, \) and \( L \) be the events that a sock is black, cotton, and long, respectively. We know:

\[
P(BL) = 0.5, \quad P(B^cL) = 0.2, \quad P(B^cL^c) = 0.2, \quad \text{and} \quad P(B^cC^c) = P(C^cL^c) = 0.
\]

\[L = (BL) \cup (B^cL)\] is a disjoint union of events, so

\[
P(L) = P(BL) + P(B^cL) = 0.5 + 0.2 = 0.7.
\]

(b) We can express the sample space as a disjoint union of events:

\[S = (BL) \cup (B^cL) \cup (BL^c) \cup (B^cL^c)\]

\[\therefore \quad 1 = P(BL) + P(B^cL) + P(BL^c) + P(B^cL^c).
\]

Solving for \( P(BL^c) \) yields

\[
P(BL^c) = 1 - 0.5 - 0.2 - 0.2 = 0.1.
\]

\[B = (BL) \cup (BL^c)\] is a disjoint union of events, so

\[
P(B) = P(BL) + P(BL^c) = 0.5 + 0.1 = 0.6.
\]

(c) By Demorgan’s Law and the fact \( P(E^c) = 1 - P(E) \), we have

\[
P(B^c \cup L) = P((BL^c)^c) = 1 - P(BL^c) = 1 - 0.1 = 0.9.
\]

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Problem 1.2

(a) For any events $A$ and $B$ in a sample space $S$, show that $P(AB) \geq 1 - P(A^c) - P(B^c)$.

(b) Consider two standard coin flips. If $A_1$ and $B_1$ are the events the first and second coin, respectively, are heads, show the inequality in (a) is not tight.

(c) Consider two standard coin flips. If $A_2$ is the event either coin is heads and $B_2$ is the event either coin is tails, show the inequality in (a) is tight.

(d) If the inequality is tight in (a), what is the relationship between sets $A$ and $B^c$?

Solutions

(a) Using DeMorgan’s Law and the fact $P(E^c) = 1 - P(E)$, we have

$$P(AB) = P((A^c \cup B^c)^c) = 1 - P(A^c \cup B^c) = 1 - (P(A^c) + P(B^c) - P(A^c B^c))$$

and $P(A^c B^c) \geq 0$, so

$$P(AB) = 1 + P(A^c B^c) - P(A^c) - P(B^c) \geq 1 - P(A^c) - P(B^c).$$

(b) For two standard coin flips, we have $S = \{HH, HT, TH, TT\}$ and

$$P(A_1^c) = P(\{TH, TT\}) = 1/2$$
$$P(B_1^c) = P(\{HT, TT\}) = 1/2$$
$$P(A_1 B_1) = P(\{HH\}) = 1/4.$$

Thus, $P(A_1 B_1) = 1/4 > 0 = 1 - P(A_1^c) - P(B_1^c)$.

(c) For two standard coin flips, we have $S = \{HH, HT, TH, TT\}$ and

$$P(A_2^c) = P(\{TT\}) = 1/4$$
$$P(B_2^c) = P(\{HH\}) = 1/4$$
$$P(A_2 B_2) = P(\{HT, TH\}) = 1/2.$$

Thus, $P(A_2 B_2) = 1/2 = 1 - P(A_2^c) - P(B_2^c)$.

(d) If $P(AB) = 1 - P(A^c) - P(B^c)$, then $P(A^c B^c) = 0$. So

$$P(B^c) = P(A^c B^c) + P(AB) = P(AB)$$

which implies $AB^c = B^c$. Therefore, $B^c \subseteq A$.

In part (c), $B_2^c = \{HH\} \subset \{HH, HT, TH\} = A_1$. 
Problem 1.3

Let $A$, $B$, and $C$ be events in the sample space $S$. Write the following probability in terms of a sum of disjoint events.

(a) $E_1$ is the event exactly one of the events occurs.

(b) $E_2$ is the event that at least one of $B$ and $C$ occurs but $A$ does not.

(c) $E_3$ is the event that $C$ occurs and exactly one of $A$ and $B$ occur.

(d) $E_4$ is the event that either only $A$ and $B$ occur or only $B$ and $C$ occur.

(e) Now suppose $P(E_2) = 0.3$, $P(E_3) = 0.2$, and $P(E_4) = 0.1$. Find $P(E_1)$.

Solutions

(a) Only $A$ occurs: $AB^cC^c$, only $B$ occurs: $A^cBC^c$, only $C$ occurs: $A^cB^cC$, so
\[ P(E_1) = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC). \]

(b) At least one of $B$ and $C$ occurs but $A$ does not:
\[ E_2 = A^c(B \cup C) = A^cBC \cup A^cBC^c \cup A^cBC^c \cup A^cB^cC^c, \] so
\[ P(E_2) = P(A^cBC) + P(A^cBC^c) + P(A^cB^cC). \]

(c) $A$ occurs but $B$ does not: $AB^c = AB^cC \cup AB^cC^c$, so
\[ P(E_3) = P(AB^cC) + P(AB^cC^c). \]

(d) $C$ occurs and exactly one of $A$ and $B$ occur: $AB^cC \cup A^cBC$, so
\[ P(E_4) = P(AB^cC) + P(A^cBC). \]

(e) We have
\[ P(E_1) = P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC) \]
\[ P(E_4) = P(AB^cC) + P(A^cBC) \]
and
\[ P(E_2) + P(E_3) = \left[ P(AB^cC^c) + P(A^cBC^c) + P(A^cB^cC) \right] + \left[ P(A^cBC) + P(AB^cC) \right] \]
\[ = P(E_1) + P(E_4) \]
Solving for $E_1$ yields
\[ P(E_1) = 0.3 + 0.2 - 0.1 = 0.4. \]